Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH762P / MTH762U: Continuous-Time Models in Finance

Duration: 3 hours

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: A. Gnedin, M. Phillips

In this exam $(B(t), t \geq 0)$ denotes the standard Brownian motion (BM), which is considered together with the natural filtration $\mathcal{F}^B = (\mathcal{F}_t^B, t \geq 0)$.

Question 1 [25 marks]. Consider the process

$$X(t) = e^{-t} + \int_0^t e^s dB(s), \quad t \geq 0.$$

(a) Is the process $(X(t),\ t \geq 0)$ Gaussian? Explain your answer. [**5**]

(b) Find the covariance function $\text{Cov}(X(s), X(t))$. [**5**]

(c) Show that the random variables $\xi = X(2)$ and $\eta = X(5) - X(3)$ are independent. [**5**]

(d) Determine $\mathbb{E}[X(t) \,|\, \mathcal{F}_s^B] - X(s)$ for $0 \leq s < t$. [**5**]

(e) Which of the following applies to $(X(t),\ t \geq 0)$: martingale, submartingale, supermartingale? Explain your answer. [**5**]

Question 2 [25 marks]. Consider the stock-bond model with fixed interest rate r and the stock price satisfying the stochastic differential equation

$$dS(t) = S(t)[\alpha dt + \sigma dB(t)], \quad t \in [0, T],$$

with some initial price $S(0)$. Consider the square-root option of European style, which at given expiration time T has value

$$V(T) = \sqrt{S(T)}.$$

Let $v(t, x)$ be the no-arbitrage price of the option at time t given the stock price $S(t) = x$.

(a) Describe the process $(\sqrt{S(t)},\ t \geq 0)$ under the risk-neutral measure, in particular determine the expected value of $\sqrt{S(t)}$. **[6]**

(b) Determine the no-arbitrage price $v(t, x)$ explicitly. **[4]**

(c) Verify that $v(t, x)$ satisfies the Black-Scholes-Merton partial differential equation. **[5]**

(d) Let $\Delta(t, x)$ be the number of shares in a self-financing stock-bond portfolio needed to hedge the option. Determine $\Delta(t, x)$ explicitly. **[4]**

(e) Now suppose the square-root option is of American style. What is then the optimal time to exercise the option? **[6]**

Question 3 [25 marks]. Consider a stock-bond market model with continuously compounded interest rate r, time horizon T and three stocks described by a system of stochastic differential equations

$$dS_i(t) = S_i(t)[\alpha_i dt + \sigma_{i1} dB_1(t) + \sigma_{i2} dB_2(t)], \quad i = 1, 2, 3,$$

where $(B_1(t), t \in [0, T])$ and $(B_2(t), t \in [0, T])$ are two independent Brownian motions under the market probability measure $\mathbb{P}$, and $r, \alpha_i, \sigma_{i1}, \sigma_{i2}$ $(i = 1, 2, 3)$ are given positive constants.

(a) State in this context the First Fundamental Theorem of Asset Pricing. [**4**]

(b) Derive a stochastic differential equation for the discounted stock price process $(e^{-rt}S_i(t),\ t \in [0, T])$. [**5**]

(c) Is the market model complete if $r = 0.1$ and $\alpha_i = 0.2,\ \sigma_{i1} = \sigma_{i2} = 1$ for $i = 1, 2, 3$? [**5**]

(d) For the remainder of the question assume that

$$\begin{aligned} \alpha_1 = 0.3;\ \sigma_{11} = 0.2;\ \sigma_{12} = 0.1 \\ \alpha_2 = 0.3;\ \sigma_{21} = 0.1;\ \sigma_{22} = 0.2 \\ \alpha_3 = 0.35;\ \sigma_{31} = 0.2;\ \sigma_{32} = 0.2 \end{aligned}$$

(i) Suppose $r = 0.15$. Describe the processes $(B_1(t),\ t \in [0, T]),\ (B_2(t),\ t \in [0, T])$ under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. [**4**]

(ii) Suppose $r = 0.2$. Construct explicitly a stock-bond trading strategy, that is an arbitrage. You may look for such a strategy with positions in stocks (in shares) given by

$$\Delta_i(t) = \frac{\delta_i}{S_i(t)},\ i = 1, 2, 3,$$

for some constants δ_i. [**7**]

Question 4 [25 marks]. Let $v(x, K, \sigma, r, T)$ be the Black-Scholes price of the European call, where x is the current price of the underlying stock. This question is concerned with pricing an *exchange option*, which gives the holder the right (but not the obligation) to exchange at maturity time T one share of one stock against one share of another stock.
Consider a market with interest rate r, and two different stocks which under probability measure $\mathbb{P}$ follow the geometric Brownian motions

$$S_1(t) = S_1(0)\exp\left((r - \frac{\sigma_1^2}{2})t + \sigma_1 B_1(t)\right) \text{ and } S_2(t) = S_2(0)\exp\left((r - \frac{\sigma_2^2}{2})t + \sigma_2 B_2(t)\right),$$

where $(B_1(t),\ t \in [0,T])$ and $(B_2(t),\ t \in [0,T])$ are independent Brownian motions. The payoff of the exchange option is

$$V(T) = (S_1(T) - S_2(T))_+.$$

(a) Is the probability measure $\mathbb{P}$ risk-neutral? **[4]**

(b) Show that the quotient

$$Q(t) = \frac{S_1(t)}{S_2(t)}$$

is a geometric Brownian motion with volatility parameter $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$. **[6]**

(c) Let $\widehat{\mathbb{P}}$ be the probability measure defined by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\frac{\sigma_2^2}{2}T + \sigma_2 B_2(T)\right).$$

Describe the processes $(B_2(t),\ t \in [0,T]), (B_1(t),\ t \in [0,T])$ and $(Q(t),\ t \in [0,T])$ under the probability measure $\widehat{\mathbb{P}}$. **[5]**

(d) Show that the no-arbitrage price of the exchange option (at time $t = 0$) is

$$S_2(0)\,\widehat{\mathbb{E}}\left[(Q(T) - 1)_+\right],$$

where $\widehat{\mathbb{E}}$ denotes the expectation under the probability measure $\widehat{\mathbb{P}}$. **[5]**

(e) Show that the no-arbitrage price of the exchange option is

$$v(S_1(0), S_2(0), \sigma, 0, T),$$

with σ as in part (b). **[5]**

End of Paper.