Queen Mary
University of London

Main Examination period 2019

MTH6141/MTH6141P: Random Processes

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: David Ellis

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Question 1. [10 marks] This question is about a Markov chain $(X_0, X_1, X_2, \ldots)$ with finite state space S and transition probabilities $(p_{i,j} : \ i, j \in S)$.

(a) Define what it means for the Markov chain $(X_0, X_1, X_2, \ldots)$ to be **irreducible**. **[2]**

(b) Define what it means for the Markov chain $(X_0, X_1, X_2, \ldots)$ to be **regular**. **[2]**

(c) Prove that if $(X_0, X_1, X_2, \ldots)$ is irreducible and $p_{k,k} > 0$ for some $k \in S$, then $(X_0, X_1, X_2, \ldots)$ is regular. **[6]**

Question 2. [28 marks] Parts (a)-(e) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where as usual, the ith row (and the ith column) of P corresponds to state i, for each $i \in \{1, 2, 3, 4, 5\}$.

(a) Explain why the Markov chain $(X_0, X_1, X_2, \ldots)$ is irreducible. **[3]**

(b) Is it regular? Justify your answer. **[3]**

(c) Explain why you can know, without doing any calculation, that this Markov chain has a unique equilibrium distribution. (You may appeal to any result in the course.) **[2]**

(d) Find the equilibrium distribution of the Markov chain. **[7]**

(e) Does the Markov chain have a limiting distribution? Justify your answer. **[5]**

Parts (f) and (g) of this question are about a Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(f) Give an argument to show that the Markov chain $(Y_0, Y_1, Y_2, \ldots)$ is regular. (You may use the result stated in Question 1 part (c), without proving it.) **[4]**

(g) It can be checked that $(1/16, 3/16, 1/4, 3/8, 1/8)$ is an equilibrium distribution for this Markov chain. Does the Markov chain have a limiting distribution? If so, what is it? Justify your answer. (You may refer to any results in the course, without proving them.) **[4]**

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Question 3. [**18 marks**] A spy is in a building with five rooms, labelled $1, 2, 3, 4, 5$. There is a door between room 1 and room 2, between room 1 and room 3, between room 2 and room 3, between room 2 and room 4, between room 3 and room 4, between room 3 and room 5 and between room 4 and room 5. (There are no other doors.) Whenever the spy is in room 1,2 or 3, he chooses uniformly at random one of the doors leading from that room, and he goes through this door into the room beyond. If the spy ever reaches room 4, he is caught in a lethal trap and dies, and if the spy ever reaches room 5, he is caught by a policeman who imprisons him in room 5.

(a) This random process can be modelled as a Markov chain with state space $\{1, 2, 3, 4, 5\}$ (with the state i corresponding to the situation where the spy is in room i, for each $i \in \{1, 2, 3, 4, 5\}$). Write down the two absorbing states. [**2**]

(b) Draw a transition graph for this Markov chain. [**5**]

(c) Suppose that the spy starts in room 1. Using first-step analysis, or otherwise, find the probability that he reaches room 4 before reaching room 5. [**8**]

(d) What is the probability that the spy eventually reaches either room 4 or room 5? Justify your answer. (You may appeal to any result in the course, without proving it.) [**3**]

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Question 4. [27 marks]

(a) State the **Superposition Lemma** for Poisson processes. [**3**]

(b) State the **Thinning Lemma** for Poisson processes. [**4**]

Parts (c)-(g) of this question are about an experiment with a particle detector, which is switched on at time 0, and then left on forever. Beta particles arrive at the detector according to a Poisson process with rate 2 per hour. Gamma particles arrive at the detector according to an independent Poisson process with rate 4 per hour. Your answers to parts (c)-(g) should be expressed in terms of powers of e (where necessary), but they should be simplified in all other ways. You may use any result in the course, but you should show your working.

(c) What is the probability that exactly 2 beta particles arrive at the detector in the first 30 minutes of the experiment? [**3**]

(d) What is the probability that the total number of particles (beta particles plus gamma particles) which arrive in the first 30 minutes, is 2? [**4**]

(e) Given that exactly 3 beta particles arrive during the first hour, what is the probability that at least one beta particle arrives during the second hour? [**4**]

(f) Given that exactly 3 beta particles arrive during the first hour, what is the probability that exactly 2 beta particles arrived during the first 20 minutes? [**5**]

(g) Suppose that the detector is faulty, and fails to detect each gamma particle (independently) with probability 1/4. What is the probability that exactly one gamma particle is detected by the particle detector, in the first hour of the experiment? [**4**]

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Question 5. [17 marks] Parts (a) and (b) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space S.

(a) Define what it means for a state $i \in S$ to be **recurrent**, in terms of the return probability $f_{i,i}$. **[2]**

(b) State a condition, in terms of the t-step transition probabilities $p_{i,i}^{(t)}$ alone, which is equivalent to the state i being recurrent. **[2]**

Parts (c)-(e) of this question are about the symmetric random walk on $\mathbb{Z}$. This is the Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\mathbb{Z}$ and (non-zero) transition probabilities given by

$$p_{i,i+1} = \tfrac{1}{2}, \quad p_{i,i-1} = \tfrac{1}{2} \quad \forall i \in \mathbb{Z}.$$

(c) Explain why

$$p_{0,0}^{(t)} = \begin{cases} \binom{t}{t/2} 2^{-t} & \text{if } t \text{ is even;} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

[5]

(d) Using parts (b) and (c), or otherwise, show that the state 0 is recurrent. You may assume the inequality

$$\binom{t}{t/2} \geq \frac{2^t}{2\sqrt{t}} \quad \text{(for any even positive integer } t\text{)},$$

and also the inequality

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \geq 2\sqrt{N-1} \quad \text{(for any positive integer } N\text{)}.$$

[5]

(e) Let $i \in \mathbb{Z}$. Is the state i recurrent or transient? Give a brief justification for your answer. (You may assume any general result in the course.) **[3]**

End of Paper.

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