Queen Mary
University of London

**Main Examination period 2020 – January – Semester A**

# MTH6106 / MTH6106P: Group Theory

**Duration: 2 hours**

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**You should attempt ALL questions. Marks available are shown next to the questions.**

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**Examiners: Matthew Fayers and Alex Fink**

In this paper, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order $n$.
- $\mathcal{U}_n$ is the set of integers between 0 and $n$ which are prime to $n$, with the group operation being multiplication modulo $n$.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $\mathrm{GL}_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

**Question 1 [21 marks].**

(a) Give the definition of a **group**. **[3]**

(b) Give the definition of a **subgroup**. **[2]**

(c) Let

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; a + c = b + d \right\}.$$

Prove that $H$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. **[5]**

Suppose $G$ is a group and $f, g \in G$.

(d) Prove that the inverse of $g$ is unique. **[4]**

(e) Give the definition of the **order** of $g$. **[2]**

(f) Suppose $g$ has order 4, and $gf = f^{-1}g$. What is the order of $fg$? *[Show your working.]* **[5]**

**Question 2 [18 marks].** Suppose $G$ is a group and $f, g \in G$.

(a) Define what it means to say that $f$ and $g$ are **conjugate** in $G$. **[2]**

(b) Give the definition of the **conjugacy class** of $g$ in $G$. **[2]**

(c) Prove that if $f$ and $g$ are conjugate, then they have the same order. **[5]**

(d) Find all the elements in the conjugacy class of $r^3$ in $\mathcal{D}_{10}$. *[Show your working.]* **[5]**

(e) Write down five different elements of $\mathcal{S}_4$ of which no two are conjugate. *[You do not need to prove anything.]* **[4]**

**Question 3 [12 marks].** Suppose $G$ is a group, $H$ is a subgroup of $G$ and $g \in G$.

(a) Define what it means to say that $H$ is **normal** in $G$. **[2]**

(b) Give the definition of the **right coset** $Hg$. **[2]**

(c) In the case where $H$ is a normal subgroup of $G$, prove that $Hg = gH$. **[4]**

(d) Now suppose $G = \mathcal{U}_{21}$ and $H = \{1, 8, 13, 20\}$. Find all the right cosets of $H$ in $G$. **[4]**

**Question 4 [17 marks].** Suppose $G$ and $H$ are groups.

(a) Give the definition of a **homomorphism** from $G$ to $H$. **[2]**

(b) Give the definition of an **automorphism** of $G$. **[2]**

(c) Give the definition of the **automorphism group** of $G$. **[2]**

(d) Find all the automorphisms of $\mathcal{C}_8$, and find the Cayley table for $\mathrm{Aut}(\mathcal{C}_8)$. *[Show your working.]* **[7]**

(e) Write down an automorphism of $\mathcal{Q}_8$ that maps $i$ to $-j$. *[You do not have to prove anything, but you should say where each element of $\mathcal{Q}_8$ maps to.]* **[4]**

 

**Question 5 [17 marks].** Suppose $G$ is a group and $X$ is a set.

(a) Give the definition of an **action** of $G$ on $X$. **[3]**

(b) Give an example of a non-trivial action of $\mathcal{D}_8$ on itself which is not transitive. *[You do not need to prove anything, but you should make it clear how your action is defined.]* **[3]**

Suppose $\pi$ is an action of $G$ on $X$, and $x \in X$.

(c) Give the definition of the **orbit** of $x$. **[2]**

(d) Give a precise statement of the **Orbit-Counting Lemma**. **[3]**

(e) Suppose we colour the vertices and edges of an equilateral triangle, and we have $n$ colours available. Say that two colourings are equivalent if one can be transformed into the other by applying a symmetry of the triangle. Use the Orbit-Counting Lemma to find the number of colourings up to equivalence. *[You should explain how you are using the Orbit-Counting Lemma as well as carrying out the calculation.]* **[6]**

**Question 6 [15 marks].** Suppose $G$ is a finite group and $p$ is a prime number.

(a) Define what it means to say that $G$ is **simple**. **[2]**

(b) Give the definition of a **Sylow $p$-subgroup of $G$.** **[2]**

(c) Find a Sylow 2-subgroup and a Sylow 3-subgroup of $\mathcal{U}_{11}$. **[4]**

(d) Give a precise statement of Sylow's Theorem 3 concerning the number of Sylow $p$-subgroups of a finite group. **[3]**

(e) Use this theorem to show that there is no simple group of order 44. **[4]**

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**End of Paper.**