M. Sc. Examination by course unit 2014

MTH6142P Complex Networks

Duration: 2 hours

Date and time: 7 May 2014 2:30pm

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): Ginestra Bianconi

Question 1

Structure and centrality measures for a given network
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 4$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

a) Is the network directed or undirected? (Give reasons) **(6 marks)**
b) Draw the network. **(6 marks)**
c) Write the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}\}$. **(8 marks)**
d) Write the in-degree distribution of the network $P^{in}(k)$ for $k = 0, 1, 2, 3$ and the out degree distribution of the network $P^{out}(k)$ for $k = 0, 1, 2, 3$. **(8 marks)**
e) The Katz centrality vector $\mathbf{x}$ has elements x_i indicating the Katz centrality of node $i = 1, 2 \ldots N$. Calculate $\mathbf{x}$ using the following definition

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1}\mathbf{1} = \beta\sum_{n=0}^{\infty}\alpha^n\mathbf{A}^n\mathbf{1}, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ and where we have indicated with $\mathbf{1}$ the column vector with elements $1_i = 1\ \forall i = 1, 2 \ldots, N$ and with $\mathbf{I}$ the $N \times N$ identity matrix. **(12 marks)**

Question 2

Giant component in random networks with given degree distribution
A random network with given degree distribution $P(k)$ has a giant component if and only if the Molloy-Reed criterion is satisfied, i.e. $\frac{\langle k^2 \rangle}{\langle k \rangle} > 2$, where $\langle \ldots \rangle$ indicates the average over the degree distribution of the network.
a) Using the properties of the generating function $G(z) = \sum_k P(k)z^k$ for a Poisson random network with degree distribution $P(k) = c^k e^{-c}/k!$ and $c > 0$ show that

- *i)* $\langle k \rangle = c$
- *ii)* $\langle k(k-1) \rangle = c^2$.

(10 marks)
b) Using the result of part a) show that that for a Poisson random network the Molloy-Reed criterion is equivalent to the following condition on the average degree: $\langle k \rangle = c > 1$. **(5 marks)**
c) Evaluate, in the continuous approximation, $\langle k \rangle$ and $\langle k^2 \rangle$ for a scale-free network of N nodes with degree distribution $P(k) = Ck^{-\gamma}$ where C is the normalization constant and the power-law exponent γ is greater than 2, i.e. $\gamma > 2$. Assume that the maximal degree K is given by $K = \min(\sqrt{N}, N^{1/(\gamma-1)})$ and the minimal degree is given by $k_{min} = 1$. **(10 marks)**
d) Show that in large N limit, scale-free networks with power-law exponent $\gamma \in (2, 3]$ always satisfy the Molloy-Reed criterion. **(5 marks)**

[Please turn over]

Question 3

The Barabasi-Albert model

The Barabasi-Albert (BA) model is the simplest growing network model that exhibits a power-law degree distribution. At time $t = 0$ the network is formed by two nodes joined by a link.

- At every time step a single new node joins the network, so that at time t there will be exactly $N = 2 + t$ nodes. Every new node has initially $m = 1$ links.
- Each new link is attached to an existing node of the network. The target node i is chosen with probability Π_i following the preferential attachment rule $\Pi_i = \frac{k_i}{\sum_{j=1}^{N} k_j}$, where k_i is the degree of the node i.

a) What is the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i in the mean-field approximation? **(10 marks)**

b) What is the degree distribution of the network at large times in the mean-field approximation? **(10 marks)**

c) Using the degree distribution obtained in part (b) and assuming that the maximal degree of the network is $K = \sqrt{t}$, calculate $\langle k^2 \rangle$ in the continuous approximation, where $\langle \ldots \rangle$ indicates the average over the degree distribution of the network. Comment on the limit of $\langle k^2 \rangle$ for $t \to \infty$. **(10 marks)**

End of Paper