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Inductive step: Prove that if $P(n-1) \Rightarrow P(n)$ for every $n \ge 2$.

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Very helpful to give a name to the statement you're trying to prove

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So by induction P(n) is true for all n.

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 $< 2^{n-1} + 2^{n-1}$ this step is clever!

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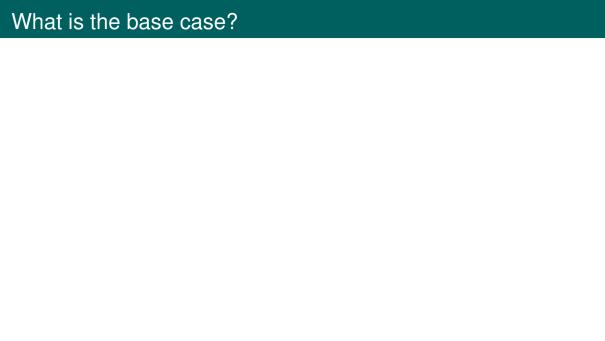
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- ▶ Work through the proof for special values of the variables, like n = 1. For an induction proof, check the case n = 2 carefully.

More guidance for proofs

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Eugenia Cheng is a British mathematician, educator, concert pianist and composer. She has several books and numerous YouTube videos explaining maths for maths students and for the general public.