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Inductive step: Prove that if $P(n-1) \Rightarrow P(n)$ for every $n \geqslant 2$.

Example of a proof by induction

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Very helpful to give a name to the statement you're trying to prove

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So by induction $P(n)$ is true for all n.

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Eugenia Cheng is a British mathematician, educator, concert pianist and composer. She has several books and numerous YouTube videos explaining maths for maths students and for the general public.