Queen Mary
University of London

MTH6104: Algebraic structures II

Duration: 2 hours

Date and time: 31st May 2013, 10:00 a.m.

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.

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Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): Matthew Fayers

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In this paper, we use the following conventions and notation:

- $\mathcal{D}_{2n}$ is the group of symmetries of a regular n-sided polygon. Its elements are

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s,$$

  where r is a clockwise rotation through $(360/n)^\circ$, and s is a reflection. The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- Q_8 is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

- $\mathrm{GL}_2(\mathbb{R})$ is the group of invertible 2×2 matrices over $\mathbb{R}$, with the group operation being matrix multiplication.

In any question, you may freely use the Coset Lemma.

Question 1.

(a) Give the definition of a **group**. **[3]**

Suppose G is a group and $g \in G$.

(b) Give the definition of the **subgroup generated by** g, and prove that it is a subgroup of G. *[You may use elementary rules for manipulating powers of elements.]* **[6]**

(c) Give the definition of the **order** of g. **[2]**

(d) Prove that $\mathrm{ord}(g) = \mathrm{ord}(g^{-1})$. *[You should include the case where* $\mathrm{ord}(g) = \infty$.*]* **[5]**

(e) Let $h = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Find $\langle h \rangle$. **[4]**

(f) Suppose $\mathrm{ord}(g) = 20$. Find elements $h, k \in G$ such that $\mathrm{ord}(h) = 4$, $\mathrm{ord}(k) = 5$, and $hk^{-1} = k^{-1}h = g$. *[You do not need to prove anything.]* **[5]**

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Question 2.

(a) Define the terms **normal subgroup** and **simple group**. *[You do not need to define what a subgroup is.]* **[4]**

(b) Suppose $g \in \mathcal{S}_n$. Explain how to write down the **disjoint cycle notation** for g. **[3]**

(c) Suppose $g, h \in \mathcal{S}_n$. How can you tell from the disjoint cycle notation whether g and h are conjugate in $\mathcal{S}_n$? **[3]**

(d) Write down two different elements g, h of $\mathcal{S}_4$ which are conjugate, and an element k such that $kgk^{-1} = h$. *[You do not need to prove anything.]* **[3]**

(e) Define the **alternating group** $\mathcal{A}_n$. **[3]**

(f) Prove that any element of $\mathcal{A}_n$ can be written as a product of 3-cycles. **[5]**

(g) Explain briefly how this result is used to prove that $\mathcal{A}_n$ is simple for $n \geqslant 5$. **[4]**

Question 3.

(a) Suppose G is a group and H a subgroup of G. Give the definition of a **right coset** of H in G, and the **index** of H in G. **[4]**

(b) Suppose $G = \mathcal{D}_{12}$, and $H = \{1, r^3, s, r^3 s\}$. Write down all the right cosets of H in G. **[5]**

(c) Suppose X is a right coset of H in G. Prove that $|X| = |H|$. **[6]**

(d) Now suppose N is a normal subgroup of G. Give the definition of the **quotient group** G/N. *[You do not have to prove that G/N is a group, but you should show that the group operation is well-defined.]* **[6]**

(e) Now suppose $G = \mathcal{S}_4$, and

$$X = \{(1\ 3), (1\ 2\ 3\ 4)\}.$$

Write down a subgroup $H \leqslant G$ and an element $g \in G$ such that $X = Hg$. *[You do not have to prove anything, but explaining your reasoning may help you to obtain marks if you make errors in calculation.]* **[4]**

Question 4. Suppose G and H are groups.

(a) Give the definition of a **homomorphism** from G to H. **[2]**

(b) Give the definition of the **image** and **kernel** of a homomorphism. **[4]**

(c) Suppose $\phi : G \to H$ is a homomorphism. Prove that

$$\frac{G}{\ker(\phi)} \cong \mathrm{im}(\phi).$$

[You may assume that $\phi(1) = 1$ *and* $\phi(g^{-1}) = \phi(g)^{-1}$ *for* $g \in G$*. You may also assume that* $\ker(\phi) \trianglelefteq G$ *and* $\mathrm{im}(\phi) \leqslant H$*.]* **[8]**

(d) Define the terms **automorphism** and **inner automorphism**. **[4]**

(e) Suppose $\phi : Q_8 \to Q_8$ is a homomorphism satisfying

$$\phi(i) = -j, \qquad \phi(j) = k.$$

Is ϕ an automorphism of Q_8? Is it an inner automorphism? Justify your answers. **[7]**

Question 5. Suppose G is a finite group.

(a) Suppose $H \leqslant G$ and $N \trianglelefteq G$. Define the set NH, and prove that it is a subgroup of G. **[6]**

(b) What is the order of NH, in terms of the orders of N, H and $N \cap H$? *[You do not need to prove anything.]* **[2]**

Now suppose p is a prime number.

(c) Give the definition of a **Sylow p-subgroup** of G. *[You do not need to define what a subgroup is.]* **[2]**

(d) Suppose P, Q are Sylow p-subgroups of G, and that $Q \trianglelefteq G$. Prove that $P = Q$. **[7]**

(e) Give a precise statement of Sylow's Theorem 3. **[3]**

(f) Using this theorem, show that there is only one group of order 85 up to isomorphism. *[You may use Lagrange's Theorem.]* **[5]**

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Question 6.

(a) Suppose G is a group and X is a set. Give the definition of an **action** of G on X. **[3]**

(b) Suppose π is an action of G on X, and $x \in X$. Give the definition of the **orbit** of x and the **stabiliser** of x. Define what it means to say that π is **transitive**. **[5]**

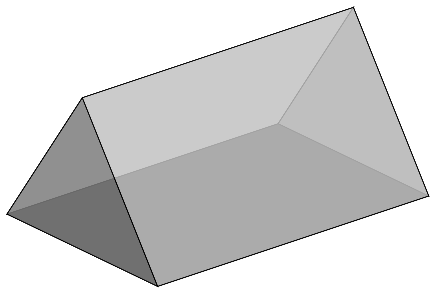
(c) Suppose $\{1\} \neq N \trianglelefteq G$. Define an action π of G on N by

$$\pi_g(n) = gng^{-1} \qquad \text{for all } g \in G, n \in N.$$

Prove that π really is an action. Is π transitive? Justify your answer. **[7]**

(d) Give a precise statement of the Orbit-Stabiliser Theorem. **[3]**

(e) Now let G be the symmetry group of a triangular prism:



What is $|G|$? Justify your answer. **[7]**

Question 7.

(a) Suppose H and K are groups. Define the **direct product** $H \times K$. *[You do not have to prove that $H \times K$ is a group, but you should say what the group operation is.]* **[3]**

(b) Suppose G is a group, and $h, k \in G$. Define the **commutator** of h and k. **[2]**

(c) Find the commutator of r and rs in $\mathcal{D}_8$. *[Show your working.]* **[3]**

(d) Now suppose H, K are normal subgroups of a group G, such that $H \cap K = \{1\}$ and $HK = G$. Using commutators, prove that $hk = kh$ for all $h \in H$ and $k \in K$. **[4]**

(e) Hence prove that $G \cong H \times K$. **[8]**

(f) Is C_9 isomorphic to $C_3 \times C_3$? Justify your answer. **[5]**

End of Paper.

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