Queen Mary
University of London

MTH6104: Algebraic structures II

Duration: 2 hours

Date and time: 24th May 2012, 10:00 a.m.

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): Matthew Fayers

In this paper, we use the following convention: $\mathcal{D}_{2n}$ is the group of symmetries of a regular n-sided polygon. Its elements are

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s,$$

where r is a clockwise rotation through $(360/n)^\circ$, and s is a reflection. The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

Question 1.

(a) Give the definition of a **group**. **[2]**

(b) Suppose $G = \{a, b, c\}$, with a binary operation given by the following table.

	a	b	c
a	b	a	c
b	a	b	c
c	c	c	b

Which of the group axioms does G satisfy? Justify your answer. **[4]**

(c) Define what is meant by the **index** of a subgroup, and state Lagrange's Theorem. **[5]**

(d) If G is a group and $f \in G$, what is meant by the **order** of f? **[2]**

Now suppose $G = \left\{1, g, g^2, g^3, g^4, g^5\right\}$ is the cyclic group of order 6.

(e) Write down the order of each element of G. **[3]**

(f) Write down a subgroup H of G containing exactly two elements. **[2]**

(g) Find all the right cosets of H. **[3]**

(h) Write down a Cayley table for G/H. **[4]**

Question 2. Write an essay on the symmetric groups and the alternating groups. *[You should include precise definitions and statements of results, illustrated by examples, and give some proofs, with outlines of longer proofs.]* **[25]**

Question 3. Suppose G is a group and X is a set.

(a) Define what is meant by an **action** of G on X. **[2]**

(b) Suppose N is a normal subgroup of G, and define $\pi_g : N \to N$ for each $g \in G$ by $\pi_g(n) = gng^{-1}$. Prove that π is an action of G on N. **[5]**

(c) Suppose π is an action of G on X, and $x \in X$. Define what is meant by the **stabiliser** and the **orbit** of x. Prove that the stabiliser of x is a subgroup of G. **[6]**

(d) Give precise statements of the Orbit-Stabiliser Theorem and the Orbit-Counting Lemma. **[6]**

(e) Suppose we colour the edges of a square, and we regard two colourings as the same if one can be transformed into the other by a symmetry of the square. If we have four colours available, how many different colourings are there? Justify your answer. **[6]**

Question 4. *[In this question, you may assume any results you need about actions of groups.]* Suppose p is a prime.

(a) What does it mean to say that a finite group is a p-**group**? **[2]**

(b) Suppose

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \middle|\ a, b, c \in \mathbb{F}_p \right\},$$

where $\mathbb{F}_p$ denotes the set $\{0, \ldots, p-1\}$ with addition and multiplication modulo p. Find a normal subgroup of G of order p, and another normal subgroup of order p^2. *[You do not have to prove anything.]* **[7]**

(c) Give the definition of a **Sylow p-subgroup**. **[2]**

(d) Prove that every finite group has at least one Sylow p-subgroup. *[You may assume that if p does not divide b, then the binomial coefficient $\binom{p^a b}{p^a}$ is not divisible by p.]* **[8]**

(e) Find a Sylow 2-subgroup of $\mathcal{D}_{12}$. Is this subgroup normal? Justify your answer. **[6]**

Question 5. Suppose G is a group.

(a) If $f, g \in G$, what does it mean to say that f and g are **conjugate**? **[2]**

(b) Give an example of a group G and elements f, g such that f^2 and g^2 are conjugate but f and g are not. *[You do not need to prove anything.]* **[3]**

(c) Prove that conjugacy is an equivalence relation on G. **[5]**

(d) Find (with proof) all the conjugacy classes in $\mathcal{D}_{10}$. *[You may assume that two conjugate elements have the same order, and you may state without proof what the orders of the elements of $\mathcal{D}_{10}$ are.]* **[6]**

(e) Give the definition of a **commutator**, and the **commutator subgroup** G'. **[4]**

(f) Suppose n is a commutator in G, and $h \in G$. Prove that $hnh^{-1} \in G'$. **[5]**

Question 6.

(a) Define what is meant by a **homomorphism** between two groups, and by an **automorphism** of a group. **[3]**

(b) Prove that the inverse of an automorphism is an automorphism. **[4]**

(c) Suppose G is a group, and define $\phi : G \to G$ by $\phi(g) = g^{-1}$. What property does G need to have for ϕ to be a homomorphism? Explain your answer. **[5]**

(d) Find (with proof) all automorphisms of C_6. **[5]**

(e) Define the terms **inner automorphism** and **outer automorphism**. **[4]**

(f) Write down an outer automorphism of $\mathcal{D}_8$. *[You should write down where each element of $\mathcal{D}_8$ maps to, but you do not need to prove anything.]* **[4]**

Question 7.

(a) Define the terms **normal subgroup** and **simple group**. *[You do not need to define what a group or a subgroup is.]* **[4]**

(b) Suppose G is a group, $H \leqslant G$ and $N \trianglelefteq G$. Define the group NH, and prove that it is a subgroup of G. **[6]**

(c) Give an example of a group G and two subgroups H, N of G such that NH is not a subgroup of G. Justify your answer briefly. **[5]**

(d) Give a precise statement of the Third Isomorphism Theorem. **[3]**

(e) Suppose G is a group and N is a normal subgroup of G such that N and G/N are simple. If M is a normal subgroup of G other than $\{1\}, N$ or G, prove that $M \cong G/N$. *[You may assume that there is no H such that $N < H \lhd G$.]* **[7]**

End of Paper.