RELATIVITY – MTH6132

PROBLEM SET 6

1. The Bondi metric, used in the study of gravitational radiation, has line element in the coordinate (u, r, θ, φ) given by

$$\begin{aligned} ds^2 &= \left(\frac{f}{r}e^{2\beta} - g^2 r^2 e^{2\alpha}\right) du^2 + 2e^{2\beta} du dr + 2gr^2 e^{2\alpha} du\, d\theta \\ &\quad - r^2\left(e^{2\alpha} d\theta^2 + e^{-2\alpha}\sin^2\theta d\varphi^2\right). \end{aligned}$$

Here, f, g, α and β are functions of (u, r, θ, φ). Using these coordinates, write down the matrix representation of the metric g_{ab}. **Hint:** First write down x^a.

2. Start with spherical coordinates $(r, \theta, \varphi))$ in $\mathbb{R}^3$ and the line element given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

(a) Given the contravariant vector $X^a = (1, r, r^2)$, find X_a.

(b) Given the covariant vector $Y_a = (0, -r^2, r^2\cos^2\theta)$, find Y^a.

3. A type (0,2) tensor is *conserved* if

$$\nabla^a T_{ab} = 0.$$

Show that if X^a satisfies the equation $\nabla_{(a} X_{b)} = 0$, and T_{ab} is symmetric, then the vector $V_a = T_{ab}X^b$ satisfies

$$\nabla^a V_a = 0.$$

4. Let $S_b{}^c$ denote a $(1, 1)$ tensor.

(a) Give the formula for the covariant derivative $\nabla_a S_b{}^c$ in terms of the connection coefficients.

(b) Show that $\nabla_a \delta_b{}^c = 0$.

(c) Show that if the dimension of the manifold is 4, then $\delta_a{}^a = 4$.

5. Using that $\nabla_a g_{bc} = 0$ and $\nabla_c \delta_a{}^b = 0$, show that $\nabla_a g^{bc} = 0$. **Hint:** how are g_{ab} and g^{ab} related to each other?

6. Consider the two-dimensional space given by

$$ds^2 = e^y dx^2 + e^x dy^2.$$

(a) Calculate the covariant and contravariant components of the metric tensor for this spacetime.

(b) Employ the formula for the Christoffel symbols (connection) given in the notes to calculate the components $\Gamma^1{}_{11}$, $\Gamma^1{}_{12}$ and $\Gamma^2{}_{11}$ of the connection. *Note the identification $(x, y) \to (x^1, x^2)$ is used here.*

7. Let X^a be the tangent vector to a geodesic. Recall the norm of this vector,

$$|X|^2 = g_{ab} X^a X^b.$$

Using the Leibniz property of the covariant derivative, show that the norm of the tangent vector is conserved along geodesics, i.e. that:

$$X^a \nabla_a \left(|X|^2\right) = 0.$$

Therefore, if a geodesic is timelike (or spacelike, or null, respectively) at one point, then it is timelike (or spacelike, or null, respectively) along its entire length.

8. V^a is called a *Killing vector* if it satisfies the following equation

$$2\nabla_{(a}V_{b)} = \nabla_a V_b + \nabla_b V_a = 0.$$

Let X^a be the tangent vector to a geodesic, let V^a be a Killing vector, and define the scalar E by

$$E \equiv V_a X^a = g_{ab} V^a X^b.$$

Show that E is conserved along geodesics, i.e. that $X^a \nabla_a E = 0$.

9. The metric for a particular 2-dimensional spacetime is given by

$$ds^2 = -e^{2Ar} dt^2 + dr^2$$

where A is an arbitrary constant. Calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric.

10. Calculate the Christoffel symbols for the metric on $\mathbb{R}^2$ in polar coordinates and use this to write down the geodesic equations. Can you solve these equations to show that geodesics on the plane must be straight lines?

11. Consider the general static spherically symmetric spacetime in four dimensions:

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)\,.$$

(a) Compute the Christoffel symbols.

(b) The components of the Riemann tensor.

(c) The components of the Ricci tensor.

12. Consider the following spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)\,,$$

where $M > 0$ is a constant.

(a) Compute the Christoffel symbols.

(b) The components of the Riemann tensor.

(c) The components of the Ricci tensor.

13. Consider the following line element:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,.$$

(a) By direct calculation or otherwise, show that the only non-vanishing Christoffel symbols are $\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = a\,a'$ and $\Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^y_{ty} = \Gamma^y_{yt} = \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{a'}{a}$, where $a' \equiv \frac{da}{dt}$. (*Hint:* note that by symmetry you only need to compute $\Gamma^t{}_{xx}$ and Γ^x_{tx}.)

(b) Show that the tt and xx components of the Ricci tensor are given by:

$$R_{tt} = -\frac{3\,a''(t)}{a(t)}\,,$$
$$R_{xx} = a(t)\,a''(t) + 2\,a'(t)^2\,.$$

Further Exploration: Recall that the transformation from Cartesian coordinates in $\mathbb{R}^3$ to spherical coordinates can be written as

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

- Letting unprimed coordinates be (x, y, z) and primed coordinates be (r, θ, φ), write down the Jacobian matrix associated with the transformation above.
- Express the basis vectors for spherical coordinates $\{e_r, e_\theta, e_\varphi\}$ as expansions in terms of the Cartesian basis $\{e_x, e_y, e_z\}$.
- What happens when we differentiate these basis vectors? Verify that

$$\frac{\partial e_r}{\partial r} = 0, \quad \frac{\partial e_r}{\partial \theta} = \frac{1}{r}e_\theta, \quad \frac{\partial e_r}{\partial \varphi} = \frac{1}{r}e_\varphi, \quad \frac{\partial e_\theta}{\partial r} = \frac{1}{r}e_\theta, \quad \frac{\partial e_\varphi}{\partial r} = \frac{1}{r}e_\varphi.$$

  Compute the remaining partial derivatives of the other two basis vectors.
- What have you actually accomplished? It turns out that when you differentiate basis vectors, you will just get an expansion in terms of the basis vectors with "weights" or weighting coefficients. These coefficients are precisely the $\Gamma^a{}_{bc}$ discussed in class!