RELATIVITY – MTH6132

PROBLEM SET 5

1. What is the transformation law for a

(a) Type $(1,0)$ tensor? Type $(0,2)$ tensor?

(b) Using part **(a)**, how does the object $T_{ab}V^b$ transform? Is it a tensor?

2. If A^a is a contravariant vector and B_a is a covariant vector, then show that A^aB_a is a scalar (hint: show that $A'^a B'_a = A^a B_a$)

3. If $T^{ab}A_aB_b$ is a scalar for any covariant vectors A_a, B_b, show that T^{ab} transforms like a $(2,0)$ tensor.

4. The parts of this problem are unrelated.

(a) Let ϕ be a scalar function. Prove that $\dfrac{\partial^2\phi}{\partial x^a \partial x^b}$ is not a type $(0,2)$ tensor.

(b) Prove that if the equation $V_{ab} = V_{ba}$ is true in a given coordinate frame, it is true in all frames (i.e. prove that it has a tensorial property).

5. (Note: the example on pages 42/43 of the notes will help you with this question). Let x^a and x'^a be two coordinate systems:

$$x^a = (x^1, x^2) := (x, y)$$
$$x'^a = (x'^1, x'^2) := (\rho, \theta)$$

defined by the relations

$$x = e^\rho \cos\theta$$
$$y = e^\rho \sin\theta.$$

These coordinates are known as *log-polar coordinates.*

(a) Compute $\frac{\partial x'^a}{\partial x^b}$ and $\frac{\partial x^a}{\partial x'^b}$ for $a, b \in \{1, 2\}$ (hint: you will have to invert the above relations to find ρ and θ in terms of x, y).

(b) Using $A'^a = \frac{\partial x'^a}{\partial x^b}A^b$, compute A'^1 and A'^2 in terms of A^1 and A^2.

(c) Using $A'_a = \frac{\partial x^b}{\partial x'^a}A_b$, compute A'_1 and A'_2 in terms of A_1 and A_2.

Note how the expressions differ: contravariant and covariant tensors are distinct geometric objects!

6. If $R^a{}_{bcd}$ is a tensor of type $(1,3)$, show that its contraction given by $R_{bd} = R^a{}_{bad}$ is a tensor of type $(0,2)$.

7. Show that if the contravariant tensor A^{ab} is symmetric and the covariant tensor B_{ab} is antisymmetric, then $A^{ab}B_{ab} = 0$.

8. Show that any general (non-symmetric) covariant tensor of rank two, T_{ab} say, can be expressed as the sum of its symmetric part, $T_{(ab)}$, and anti-symmetric part, $T_{[ab]}$. Hence prove that $g^{ab}T_{ab} = g^{ab}T_{(ab)}$, where g^{ab} is a general metric tensor.

9. Starting from the Minkowski line element in Cartesian coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

show that in spherical coordinates $x = r\sin\theta\sin\varphi$, $y = r\sin\theta\cos\varphi$, $z = r\cos\theta$, the line element is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2.$$

Hint: Start with Problem Set 3, Exercise #8.

10. Consider a one-form A_a and the associated object

$$F_{ab} \equiv 2\,\partial_{[a}A_{b]} = \partial_a A_b - \partial_b A_a\,.$$

(a) Is F_{ab} a $(0,2)$ tensor?

(b) Consider a rank 2 anti-symmetric tensor B_{ab}, i.e., $B_{ab} = B_{[ab]}$. Compute $\nabla_{[a}B_{bc]}$. Is $\partial_{[a}B_{bc]}$ a tensor?

11. Consider the following line element:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,. \tag{1}$$

(a) By direct calculation or otherwise, show that the only non-vanishing Christoffel symbols are $\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = a\,a'$ and $\Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^y_{ty} = \Gamma^y_{yt} = \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{a'}{a}$, where $a' \equiv \frac{da}{dt}$. (*Hint:* note that by symmetry you only need to compute $\Gamma^t{}_{xx}$ and Γ^x_{tx}).

(b) Consider the following vector field V^a:

$$V = \sqrt{1 + \frac{k^2}{a(t)^2}}\frac{\partial}{\partial t} + \frac{k}{a(t)^2}\frac{\partial}{\partial x}$$

where k is a constant and $a(t)$ is the same function as in (1). Equivalently, in components,

$$V^a = \left(\sqrt{1 + \frac{k^2}{a(t)^2}}, \frac{k}{a(t)^2}, 0, 0\right)\,. \tag{2}$$

Is this vector tangent to a geodesic?

(c) Consider the following the tensor

$$K_{ab} = a(t)^2(g_{ab} + U_a\, U_b)$$

where $U^a = (\partial_t)^a$ is the vector tangent to the t-direction (i.e., $U^a = (1, 0, 0, 0)$), g_{as} is the spacetime metric (1) and $a(t)$ is the same function as in (1). Compute the parallel transport of the quantity $K_{bc}V^bV^c$ along a curve with tangent V^a.