Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH744U / MTH744P: Dynamical Systems

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Only approved non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: D.K. Arrowsmith, I. Morris

Question 1 [30 marks]. **One dimensional systems on $\mathbb{R}$ and $\mathbb{S}$**

(a) Investigate the fixed points of each of the following systems on $\mathbb{R}$ and sketch their phase portraits.

(i) $\dot{x} = \sin(x) - \sinh(x)$, **[5]**

(ii) $\dot{x} = \cos(x)(1 - \cos(x))$. **[5]**

(b) Find a function $f : \mathbb{R} \to \mathbb{R}$, such that the phase portrait of the system $\dot{x} = f(x)$ is consistent with the xt-plane solution curve plots sketched in Fig. 1. **[5]**

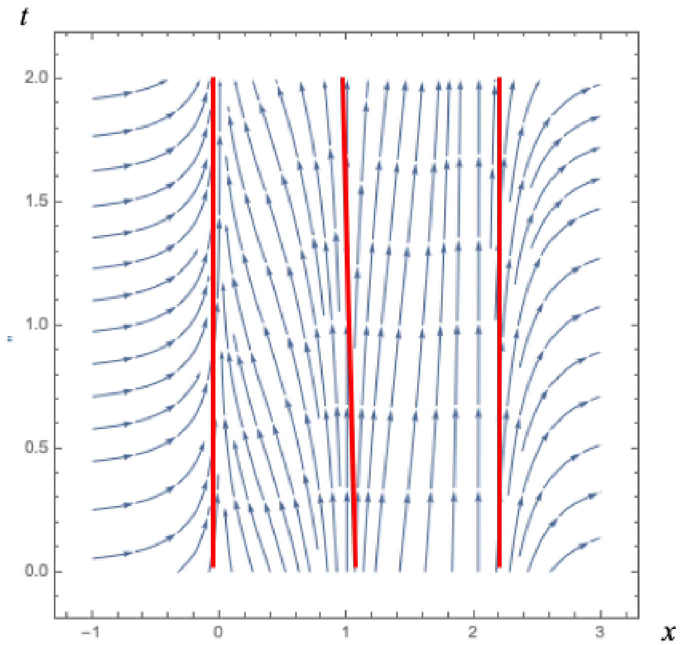

Figure 1

(c) Find an equation $\dot{x} = f(x)$ whose phase portrait is qualitatively the same as that sketched in Fig. 2 **[5]**

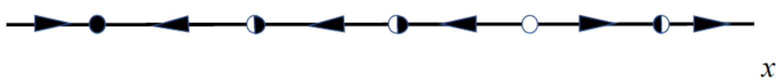

Figure 2

(d) Consider the system $\dot{x} = f(x)$ where the differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) > 0$ for $x > X_0 \in \mathbb{R}$. Show that if the system has a fixed point, then at least one of one of them is unstable. **[5]**

(e) Consider the solution curves shown in Fig. 3 for the system $\dot{\theta} = f(\theta)$, $\theta \in \mathbb{S}$, in the θt-plane with the lines $\theta = 0$ and $\theta = 2\pi$ identified. Find a function $f : \mathbb{S} \to \mathbb{R}$ such that the solution curves of $\dot{\theta} = f(\theta)$ are topologically compatible with those of Fig. 3. **[5]**

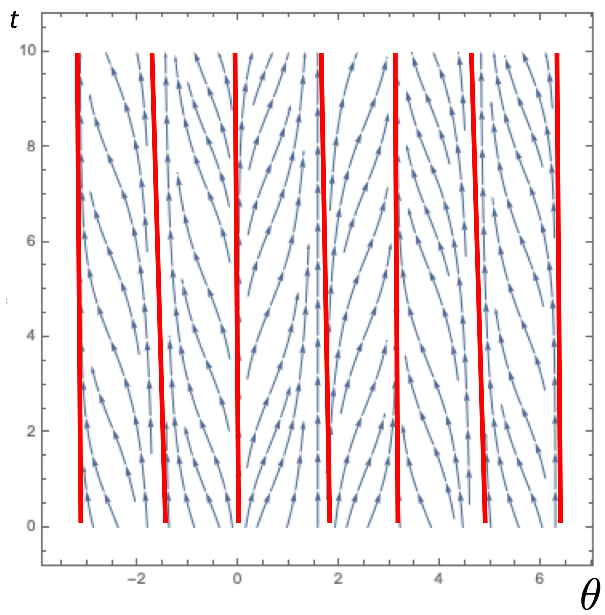

Figure 3

Question 2 [34 marks]. Bifurcations on the line

(a) Consider the dynamical system on the line $\mathbb{R}$, given by the ordinary differential equation

$$\dot{x} = 1 + rx - x^2 - rx^3, \tag{1}$$

which depends on the real parameter r.
For system (1),

(i) Investigate and display the set of fixed points in the xr-plane. **[6]**

(ii) identify potential bifurcation points $(x, r) = (x^*, r^*)$ with $r \neq 0$ for the system (1), **[6]**

(iii) Use appropriate Taylor expansions to reduce the equation to normal from at each of the bifurcation points, and hence classify their bifurcation types. **[6]**

(iv) Illustrate on the xr-plane the complete set of different qualitative types of phase portrait that can occur as the parameter $r \in \mathbb{R}$ varies. **[6]**

(v) In one sentence, describe what happens to the fixed points of the system as $r \to 0$. **[4]**

(b) Consider the dynamical system on the line $\mathbb{R}$, given by the ordinary differential equation

$$\dot{x} = Ar + Bx^2 + Cxr \tag{2}$$

which depends on the real parameter r. Why is $(x, r) = (0, 0)$ a potential bifurcation point?
When A, B, C are all non-zero, calculate the non-singular changes of variable x and parameter r which reduce Eqn (2) to the form

$$\dot{y} = \nu + y^2. \tag{3}$$

[6]

Question 3 [36 marks]. **Two-dimensional systems**

Consider the system

$$\dot{x} = y \quad , \quad \dot{y} = x - x^3 + y \,, \tag{4}$$

where $(x, y) \in \mathbb{R}^2$.

(a) Compute the fixed points of the two dimensional dynamical system given by (4). For each fixed point perform a linear stability analysis and classify the type of fixed point. Where possible, find the eigenvectors corresponding to real eigenvalues. **[8]**

(b) Sketch the flow in the phase plane in a small neighbourhood of each fixed point. **[9]**

(c) Construct a nullcline diagram for this system as follows:

(i) Compute the nullclines of the system of differential equations. **[2]**

(ii) Sketch the nullclines in the phase plane. **[4]**

(iii) The nullclines partition the phase plane into different regions. For each region, and on each nullcline, indicate the direction of the flow. **[4]**

(d) Using the results from parts a) - c), sketch the full phase portrait of the two dimensional system. The phase portrait should be consistent with the diagram produced in part c). If the system has a saddle fixed point then sketch the form of its unstable and stable manifolds in the phase portrait. **[9]**

End of Paper.