Queen Mary University of London

Main Examination period 2021 – January – Semester A

MTH744U/MTH744P : Dynamical Systems

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **4 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **3 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: D.K. Arrowsmith, C. Beck

Question 1 [28 marks]. One dimensional systems on $\mathbb{R}$ and $\mathbb{S}$

(a) Consider the following dynamical systems on the line $\mathbb{R}$, given by the ordinary differential equations

(i) $\dot{x} = x^2(x^6 + x^3 - 1)$,

(ii) $\dot{x} = \exp(-x) - \tanh(x)$,

(iii) $\dot{x} = \sin(x) - \tanh(x)$.

Investigate each system, and deduce the phase portrait on $\mathbb{R}$ for each system. **[12]**

(b) Sketch the phase portrait of a flow on the circle, $\mathbb{S}$, which has exactly 4 fixed points: one being linearly stable; one being linearly unstable; and two being saddle-nodes. Identify the basin of attraction of each of the fixed points in the phase portrait diagram. **[6]**

(c) Find a differentiable function $f : \mathbb{S} \to \mathbb{R}$ such that the differential equation $\dot{\theta} = f(\theta)$ generates the phase portrait described in (b). **[6]**

(d) Explain in words the likely structure of a phase portrait given by $\dot{\theta} = f^2(\theta)$, for **any** differentiable function $f : \mathbb{S} \to \mathbb{R}$. **[4]**

Question 2 [32 marks]. Bifurcations on the line

(a) Consider the dynamical system on the line $\mathbb{R}$, given by the ordinary differential equation

$$\dot{x} = x(1 + rx + x^2), \tag{1}$$

which depends on a real parameter r.

(i) Show that there are at most three fixed points of the system (1) for any given $r \in \mathbb{R}$ and show their location in the xr-plane. **[6]**

(ii) Locate all bifurcation points for system (1) in the xr-plane. Classify, each one as of saddle-node, transcritical or pitchfork type, by reducing to normal form. **[10]**

(iii) Indicate the different qualitative types of phase portrait for system (1) that can occur for $r \in \mathbb{R}$. **[6]**

(b) Consider the dynamical system on the line $\mathbb{R}$, given by the ordinary differential equation

$$\dot{x} = x(1 + rx - x^2), \tag{2}$$

which depends on a real parameter r.

(i) Locate the fixed points in the xr-plane for system (2); **[6]**

(ii) Sketch the different qualitative types of phase portrait which can occur. Does the system exhibit any bifurcations? **[4]**

Question 3 [40 marks]. Two dimensional systems

(a) Consider the system,

$$\dot{x} = x(1 - 2y) \quad \dot{y} = -y(1 - x). \tag{3}$$

(i) Find the fixed points of system (3) and classify them, sketch the null-clines and the vector field for the positive quadrant. **[6]**

(ii) For which fixed points of system (3) does the Hartman-Grobman theorem assist in identifying their types of stability? **[4]**

(iii) Find a first integral for system (3) of the form $f(x) + g(y) = C$, a constant. By examining the form of each of the functions $f(x), g(y)$, or otherwise, determine the nature of all the fixed points and sketch the phase portrait in the first quadrant. **[6]**

(b) Consider the system of differential equations

$$\dot{x} = x(1 - 3x^2 - y^2) - y(1 + x), \ \dot{y} = y(1 - 3x^2 - y^2) + 3x(1 + x). \tag{4}$$

(i) Compute the fixed points of the system (4). For each fixed point determine the stability using linear stability analysis. **[8]**

(ii) Consider the quantity $L = (1 - 3x^2 - y^2)^2$. Show that $\frac{dL}{dt} \leq 0$. When does $\frac{dL}{dt} = 0$? **[6]**

(iii) Using the results of part (b)(ii), or otherwise, show that the system (4) has a unique limit cycle. Is the limit cycle stable or unstable? *Give reasons for your answer.* **[6]**

(iv) Using the results of part (b)(i-iii), or otherwise, sketch the phase portrait of the system (4). **[4]**

End of Paper.