Main Examination period 2018

MTH6140: Linear Algebra II (Solutions)

Duration: 2 hours

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Question 1.

(a) (i) The list $(v_1, \ldots, v_n)$ is *linearly independent* if, for all $c_1, \ldots, c_n \in \mathbb{K}$, it is the case that $c_1v_1 + \cdots c_nv_n = \mathbf{0}$ implies $c_1 = \cdots = c_n = 0$; (ii) it is *spanning* if every vector $v \in V$ may be expressed in the form $v = c_1v_1 + \cdots c_nv_n$ for some $c_1, \ldots, c_n \in \mathbb{K}$; (iii) it is a *basis* if it is both linearly independent and spanning.

(b) (i) Yes, (ii) no, (iii) no, and (iv) yes.

(c) The span is

$$\langle u_1, \ldots, u_r \rangle = \{a_1u_1 + a_2u_2 + \cdots + a_ru_r : a_1, a_2, \ldots, a_r \in \mathbb{K}\}.$$

(d) Since the list $u_1, \ldots, u_r$ is not spanning, there is a vector in V that is not in the span $\langle u_1, \ldots, u_r \rangle$. Following the hint, let $u_{r+1} \in V$ be such a vector. We claim that the extended list $u_1, \ldots, u_r, u_{r+1}$ is linearly independent. Suppose not; then there exist $a_1, \ldots, a_{r+1} \in \mathbb{K}$, not all zero, such that $a_1u_1 + \cdots a_ru_r + a_{r+1}u_{r+1} = \mathbf{0}$. Since the list $u_1, \ldots, u_r$ is linearly independent, we must have $a_{r+1} \neq 0$. Dividing through by a_{r+1}, we obtaining an expression for u_{r+1} as a linear combination of $u_1, \ldots, u_r$. But this contradicts the choice of u_{r+1}.

(e) Repeatedly extend the list as in part (d) until the resulting list is spanning. (This must occur eventually, as V is finite dimensional.) The resulting list is spanning and linearly independent and hence a basis.

Parts (a, c–e) are bookwork; part (b) contains easy tests of understanding of basic definitions.

Question 2.

(a) The matrices are, respectively,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) (i) $\det(A)$ is unchanged, (ii) the absolute value of $\det(A)$ is unchanged but its sign is inverted, and (iii) $\det(A)$ is multiplied by c.

(c) Elementary row operations correspond to multiplication on the left by an elementary matrix. From part (b), multiplying a matrix A on the left by elementary matrix P_i has the effect of multiplication $\det(A)$ by a scalar, say $c_i \in \mathbb{K}$. Thus

$$\det(A) = \det(P_t \cdots P_1 I) = c_1 c_2 \dots c_t \det(I) = c_1 c_2 \dots c_t, \quad \text{and}$$
$$\det(AB) = \det(P_t \cdots P_1 B) = c_1 c_2 \dots c_t \det(B).$$

It follows that $\det(AB) = \det(A)\det(B)$.

(d) With A and B as above,

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \\ &= \det(A)\det(P^{-1})\det(P) = \det(A)\det(I) \\ &= \det(A). \end{aligned}$$

Parts (b–d) are bookwork; (a) is an immediate consequence of bookwork.

Question 3.

(a) $\text{Ker}(\alpha) = \{v \in V : \alpha(v) = \mathbf{0}\}$ and $\text{Im}(\alpha) = \{\alpha(v) : v \in V\}$.

(b) $\dim(\text{Ker}(\alpha) + \text{Im}(\alpha)) + \dim(\text{Ker}(\alpha) \cap \text{Im}(\alpha)) = \dim(\text{Ker}(\alpha)) + \dim(\text{Im}(\alpha))$.

(c) π is a projection on V if it is a linear map on V and $\pi^2 = \pi$.

(d) (i) yes, (ii) no, (iii) no and (iv) yes.

(e) Suppose $v \in \text{Ker}(\pi) \cap \text{Im}(\pi)$. Since $v \in \text{Ker}(\pi)$ we have $\pi(v) = \mathbf{0}$. Since $v \in \text{Im}(\pi)$ we also have $\pi(v) = v$. Putting these together, $v = \mathbf{0}$.

(f) From part (e), we have $\dim(\text{Ker}(\pi) \cap \text{Im}(\pi)) = 0$. Then, from part (b),

$$\begin{aligned}\dim(\text{Ker}(\pi) + \text{Im}(\pi)) &= \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)) - \dim(\text{Ker}(\pi) \cap \text{Im}(\pi)) \\ &= \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)).\end{aligned}$$

Parts (a–c) are bookwork; (e) is a step in the proof of a theorem in the course, so is bookwork but needs to be recognised as such; (d) just requires applying the definition, and (f) is an easy deduction.

Question 4.

(a) The characteristic polynomial of a matrix A is defined to be $p_A(x) = \det(xI - A)$.

(b) The Cayley-Hamilton Theorem states that $p_A(A) = O$ for all matrices A, where O is the zero matrix.

(c) The minimal polynomial of A is the monic polynomial $m_A(x)$ of smallest degree such that $m_A(A) = 0$.

(d) The characteristic polynomial $p_\alpha(x)$ has no repeated factors, so the minimal polynomial is also $(x-1)(x^2+1)$. Not all the factors are linear, so α is not diagonalisable.

(e) Over $\mathbb{C}$ we have $p_\alpha(x) = (x-1)(x-i)(x+i) = m_\alpha(x)$. The minimal polynomial is a product of distinct linear factors, so α is diagonalisable.

(f) For α to be diagonalisable, we must have $m_\alpha(x) = x - 1$. (There can be no repeated factor.) So any α other than the identity is not diagonalisable. An example is the linear map represented by the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The identity map is clearly diagonalisable.

Parts (a–c) are bookwork; parts (d–f) are deductions from basic results/tests of understanding, with (f) being definitely harder.

Question 5.

(a) The adjoint $\alpha^* : V \to V$ is the unique linear map satisfying $v \cdot \alpha^*(w) = \alpha(v) \cdot w$, for all $v, w \in V$. The linear map α is self-adjoint if $\alpha^* = \alpha$.

(b) Subspaces U and W are orthogonal if $u \cdot w = 0$ for all $u \in U$ and $w \in W$.

(c) The orthogonal complement of U is $U^\perp = \{v \in V : v \cdot u = 0, \text{ for all } u \in U\}$.

(d) Let $u \in U$ be arbitrary. By definition, $u \cdot v = 0$. Then

$$\alpha(u) \cdot v = u \cdot \alpha^*(v) = u \cdot \alpha(v) = u \cdot (\lambda v) = \lambda(u \cdot v) = 0,$$

where we use the facts that α is self-adjoint and v is a eigenvalue of α with eigenvalue λ. Thus, $\alpha(u)$ is orthogonal to v and hence in U.

(e) The proof is by induction on the dimension of V. The first step of the proof is to show the existence of an eigenvector v of α with eigenvalue λ (say). As in part (d), let U be the subspace of all vectors orthogonal to v. From (d) we know that the restriction of α to U is a linear map on U, which is also self-adjoint. By the induction hypothesis, there is a basis of U consisting of orthogonal eigenvectors of α. Augmenting this basis with v yields an orthogonal basis for V.

Parts (a–c) are bookwork; part (d) is (disguised) bookwork, provided it is recognised as a step in the proof of the Spectral Theorem; part (e) tests whether the student understands the proof of the Spectral Theorem "in essence".

End of Paper.