Queen Mary
University of London

Main Examination period 2017

MTH6140: Linear Algebra II (Solutions)

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Only non-programmable calculators that have been approved from the college list of non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Question 1.

(a) (i) No, (ii) yes, (iii) yes, and (iv) no.

(b) A subset U of V is a subspace of V if it is non-empty, closed under vector addition, and closed under multiplication by arbitrary scalars.

(c) Following (b), suppose that $v, v' \in U \cap W$. Since $v, v' \in U$ we know that $v + v' \in U$; similarly, $v + v' \in W$. Thus $v + v' \in U \cap W$ and $U \cap W$ is closed under vector addition. Now suppose $v \in U \cap W$ and $c \in \mathbb{K}$. Since $v \in U$ we know that $cv \in U$; similarly $cv \in W$. Thus $cv \in U \cap W$ and $U \cap W$ is closed under scalar multiplication. Finally, $U \cap W \neq \emptyset$ since $\mathbf{0} \in U \cap W$.

(d) The sum of U and W is defined by $U + W = \{u + w : u \in U \text{ and } w \in W\}$. The dimensions of the subspaces are related by

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

(e) The given spanning sets for U and W are clearly independent (one vector is not a multiple of the other) so they are bases. This $\dim(U) = \dim(W) = 2$. The vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are clearly independent and hence form a basis for $\mathbb{R}^3$; they are contained in $U + W$ and hence $U + W = \mathbb{R}^3$ and $\dim(U + W) = 3$. By part (d), $\dim(U \cap W) = 2 + 2 - 3 = 1$.

Notes. (a,b,d) are easy tests of basic concepts/results. (c) is bookwork. (e) is a routine application.

Question 2.

(a) The sign of π is $\mathrm{sign}(\pi) = (-1)^{n-c(\pi)}$, where $c(\pi)$ is the number of cycles of π. Then

$$\det(A) = \sum_{\pi} \mathrm{sign}(\pi) a_{1,\pi(1)} \dots a_{n,\pi(n)}.$$

where the sum is over all permutations of $\{1,\dots,n\}$.

(b) Let $A = (a_{i,j})$, and A' be as A, but with the first row replaced by

$$(a'_{1,1}, a'_{1,2}, \dots, a'_{1,n}).$$

Thus, B has first row

$$(a_{1,1} + a'_{1,1}, a_{1,2} + a'_{1,2}, \dots, a_{1,n} + a'_{1,n}).$$

Then

$$\begin{aligned}\det(B) &= \sum_{\pi \in S_n} \mathrm{sign}(\pi)\,(a_{1,\pi(1)} + a'_{1,\pi(1)}) a_{2,\pi(2)} \cdots a_{n,\pi(n)} \\ &= \sum_{\pi \in S_n} \mathrm{sign}(\pi)\, a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)} + \sum_{\pi \in S_n} \mathrm{sign}(\pi)\, a'_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)} \\ &= \det(A) + \det(A').\end{aligned}$$

(c) D2 is the property that the determinant of any matrix with a repeated row is 0. Property D3 is that the determinant of the identity matrix is 1.

(d) By a process of elimination, it is property D2. For example, consider the matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $\det'(A) = 1$ (there is just one term in the formula, corresponding to π being the identity), and yet A has equal rows.

Parts (a) and (c) recall basic definition; (b) is bookwork; (d) is unseen (though an exercise in the course asked a similar question about the permanent of a matrix).

Question 3.

(a) $\mathrm{Ker}(\alpha) = \{v \in V : \alpha(v) = \mathbf{0}\}$ and $\mathrm{Im}(\alpha) = \{\alpha(v) : v \in V\}$.

(b) π is a projection on V if it is a linear map on V and $\pi^2 = \pi$.

(c) Let $v \in V$ be arbitrary. Following the hint, write v as $u + w$ where $u = v - \pi(v)$ and $w = \pi(v)$. It is clear that $w \in \mathrm{Im}(\pi)$. Also, since $\pi(v - \pi(v)) = \pi(v) = \pi(\pi(v)) = \pi(v) - \pi(v) = \mathbf{0}$, we see that $(v - \pi(v)) \in \mathrm{Ker}(\pi)$.

(d) Suppose $v \in \mathrm{Ker}(\pi) \cap \mathrm{Im}(\pi)$. Since $v \in \mathrm{Ker}(\pi)$ we have $\pi(v) = 0$. Also, since $v \in \mathrm{Im}(\pi)$, we have $\pi(v) = v$. Putting these two facts together, $v = \pi(v) = \mathbf{0}$,

(e) The fact that $(I - \pi)$ is a projection follows from the chain of inequalities

$$\begin{aligned}(I-\pi)^2(v) &= (I-\pi)(v-\pi(v)) = (I-\pi)(v) - (\pi(v) - \pi(\pi(v))) \\ &= (I-\pi)(v) - (\pi(v) - \pi(v)) = (I-\pi)(v).\end{aligned}$$

(f) We have

$$v \in \mathrm{Ker}(I-\pi) \iff (I-\pi)(v) = \mathbf{0} \iff \pi(v) = v \iff v \in \mathrm{Im}(\pi),$$

and

$$v \in \mathrm{Im}(I-\pi) \iff v = (I-\pi)(v) \iff v = v - \pi(v) \iff \pi(v) = \mathbf{0} \iff v \in \mathrm{Ker}(\pi)$$

(or substitute $(I - \pi)$ for π in the previous identity).

Parts (a) and (b) recall basic definitions; (c) and (d) are bookwork; (e) and (f) did not appear in this form in the course and require some engagement with the material.

Question 4.

(a) The characteristic polynomial of a matrix A is defined to be $p_A(x) = \det(xI - A)$. The minimal polynomial of A is the monic polynomial $m_A(x)$ of smallest degree such that $m_A(A) = 0$. Such a polynomial exists by the Cayley-Hamilton Theorem.

(b) A is diagonalisable if $m_A(x)$ is a product of distinct linear factors.

(c)

$$p_A(x) = \begin{vmatrix} x-3 & 0 & 0 \\ 2 & x-1 & 1 \\ -1 & -1 & x-3 \end{vmatrix} = (x-3)\begin{vmatrix} x-1 & 1 \\ -1 & x-3 \end{vmatrix} = (x-3)(x-2)^2.$$

The minimal polynomial divides $p_A(x)$ and has the same set of roots; thus either $m_A(x) = (x-3)(x-2)$ or $m_A(x) = (x-3)(x-2)^2$. Observe that

$$m_A(A) = (A-3I)(A-2I) = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \neq O,$$

eliminating the first possibility. So the only remaining possibility is $m_A(x) = (x-3)(x-2)^2$.

(d) A is not diagonalisable, since $m_A(x)$ has a repeated factor.

Notes. Parts (a) and (b) are basic definitions, results; (c) and (d) are applications.

Question 5.

(a) $\alpha(v) = \lambda v$ and $v \neq \mathbf{0}$. $E(\lambda, \alpha) = \{v \in V : \alpha(v) = \lambda v\}$.

(b) $V = U + W$ (or $V = U \oplus W$) and $u \cdot w = 0$ for all vectors $u \in U$ and $w \in W$.

(c) $v \cdot \alpha(w) = \alpha(v) \cdot w$ for all $v, w \in V$.

(d) The eigenspaces of a self-adjoint linear map form an orthogonal decomposition of V. That is, V is the direct sum of the eigenspaces, and the eigenspaces are pairwise orthogonal.

(e) The matrix is symmetric (represents a self-adjoint linear map with respect to an orthonormal basis) and is hence diagonalisable by the Spectral theorem.

(f) The trace of a matrix $A = (a_{i,j})$ is the sum $\sum_i a_{i,i}$ of its diagonal entries.

(g) A is orthogonally similar to a diagonal matrix whose diagonal entries are the eigenvalues of A. Similar matrices have equal trace; thus the trace of A is the sum of its eigenvalues. Since A has trace 18, the missing eigenvalue is $18 - (9 - 27) = 36$.

Parts (a–d) and (f) are standard definitions and results; (e) and (g) are simple applications.

End of Paper.