Queen Mary
University of London

Main Examination period 2017

MTH6140: Linear Algebra II

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Jerrum, M. Walters

Question 1. [20 marks]
In this question, V is a vector space over a field $\mathbb{K}$.

(a) Which of the following operations are valid:

(i) adding a vector to a scalar?

(ii) multiplying a vector by a scalar?

(iii) multiplying two scalars?

(iv) multiplying two vectors? [**4**]

(b) Suppose U is a subset of V. Give easy-to-test conditions for U to be a subspace of V. [**3**]

(c) Prove that the intersection of two subspaces U and W of V is also a subspace of V. [**4**]

(d) Define the **sum** $U+W$ of two subspaces of V. State without proof a relationship between $\dim(U \cap W)$, $\dim(U)$, $\dim(W)$ and $\dim(U+W)$. [**4**]

(e) Let V be $\mathbb{R}^3$ and let U and W be the subspaces

$$U = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle \quad \text{and} \quad W = \left\langle \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\rangle.$$

Determine $\dim(U \cap W)$, $\dim(U)$, $\dim(W)$ and $\dim(U+W)$, briefly justifying your answer. [**5**]

Question 2. [20 marks]
In this question, A, A' and B are $n \times n$ matrices over a field $\mathbb{K}$.

(a) Define the **sign** of a permutation π on $\{1, \ldots, n\}$, and write down the Leibniz (sum-over-permutations) formula for the determinant of A. **[6]**

(b) Suppose that A, A' and B agree on all rows except the first. Furthermore, suppose that the first row of B is equal to the sum of the first row of A and the first row of A'. Using the formula from part (a), prove that $\det(B) = \det(A) + \det(A')$. **[6]**

(c) The identity in part (b) of this question is a special case of a property we labelled D1 in the course. State two other properties, D2 and D3 of the determinant function that together with D1 characterise the determinant (i.e., any function from $n \times n$ matrices A to $\mathbb{K}$ satisfying D1–D3 is in fact the determinant of A). **[4]**

(d) Consider the function $\det'$ from $n \times n$ matrices to $\mathbb{K}$ defined as follows: $\det'(A)$ is given by the formula of part (a) but with the summation restricted to **even** permutations, i.e., permutations π with $\text{sign}(\pi) = +1$. One of the properties D1–D3 fails for the modified function $\det'$. Which is it and why? **[4]**

Question 3. [20 marks]

(a) Suppose V is a vector space and $\alpha : V \to V$ is a linear map on V. Define the **kernel** $\text{Ker}(\alpha)$ and **image** $\text{Im}(\alpha)$ of α. **[4]**

(b) Define what it means for a linear map $\pi : V \to V$ to be a **projection** on V. **[3]**

(c) Let π be a projection on V. By considering the identity $v = (v - \pi(v)) + \pi(v)$, prove that $V = \text{Ker}(\pi) + \text{Im}(\pi)$. **[3]**

(d) With π as in part (c), prove that $\text{Ker}(\pi) \cap \text{Im}(\pi) = \{\mathbf{0}\}$. **[3]**

(e) Consider the linear map $I - \pi$ on V where I is the identity map and π is a projection. Prove that $I - \pi$ is a projection. **[3]**

(f) Prove that $\text{Ker}(I - \pi) = \text{Im}(\pi)$ and $\text{Im}(I - \pi) = \text{Ker}(\pi)$. **[4]**

Question 4. [20 marks]
In this question, A is a square matrix with entries in a field $\mathbb{K}$.

(a) Define the **characteristic polynomial** $p_A(x)$ and the **minimal polynomial** $m_A(x)$ of A. **[6]**

(b) State without proof a condition for A to be diagonalisable in terms of the minimal polynomial of A. **[3]**

(c) Compute the characteristic polynomial and minimal polynomials of the real matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$ **[8]**

(d) Is the matrix A from part (c) diagonalisable? Briefly justify your answer. **[3]**

Question 5. [20 marks]
In this question, α is a linear map on a real inner product space V.

(a) State the condition for a vector $v \in V$ to be an **eigenvector** of α with **eigenvalue** λ. Define the **eigenspace** $E(\lambda, \alpha)$. **[4]**

(b) Explain what it means for subspaces U and W of V to form an **orthogonal decomposition** of V. **[3]**

(c) Define what it means for α to be **self-adjoint**. **[2]**

(d) State a theorem (a version of the Spectral Theorem) about the eigenspaces of a self-adjoint linear map. **[4]**

(e) Consider the matrix

$$A = \begin{bmatrix} 5 & -20 & 22 \\ -20 & 17 & 2 \\ 22 & 2 & -4 \end{bmatrix}.$$

Is A diagonalisable? (This part requires no calculation, but you should justify your answer.) **[2]**

(f) Define the **trace** of a square matrix. **[2]**

(g) Suppose A is as in part (e). Given that two of the eigenvalues of A are 9 and -27, what is the third? (This part requires very little calculation.) **[3]**

End of Paper.