Main Examination period 2021 – January – Semester A

MTH6140: Linear Algebra II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: S. Majid, M. Fayers

In your answers, all expressions should be simplified as much as possible.

Question 1 [20 marks]. Let V denote the space of polynomials in x of degree ≤ 3 over a field $\mathbb{K}$. Let

$$v_1 = 1 + x + x^2 + x^3, \quad v_2 = 1 + 2x + 3x^2, \quad v_3 = 2 + 6x, \quad v_4 = 6.$$

(a) Are the vectors v_1, v_2, v_3, v_4 linearly independent over each of the following $\mathbb{K}$?

(i) $\mathbb{K} = \mathbb{R}$; **[5]**

(ii) $\mathbb{K} = \mathbb{F}_3$, the field of integers mod 3; **[5]**

(iii) $\mathbb{K} = \mathbb{F}_5$, the field of integers mod 5. **[4]**

Justify your answers.

(b) In which of the cases (i)-(iii) do v_1, v_2, v_3, v_4 span V? Justify your answers. **[6]**

Question 2 [20 marks].

(a) Prove that each type of elementary row operation on an $m \times n$ matrix is invertible by giving the inverse operation. **[4]**

(b) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \end{bmatrix}$ over the field $\mathbb{Q}$ of rationals. Using elementary row and column operations, or otherwise, find invertible matrices P, Q such that the matrix PAQ is in canonical form for equivalence. **[6]**

(c) Let A be an $n \times n$ matrix with a **column** of zeros. Show **directly** from the Leibniz formula for determinants given in Lectures that $\det(A) = 0$. **[6]**

(d) Now suppose that V has a basis v_1, v_2, v_3 and W a basis w_1, w_2, and let $\alpha : V \to W$ be the linear map corresponding to the matrix A in (b) with respect to these bases. State the values $\alpha(v_1), \alpha(v_2), \alpha(v_3)$. What is the rank $\rho(\alpha)$? **[4]**

Question 3 [20 marks].

(a) Let V be a vector space over a field $\mathbb{K}$. Define what is meant by a **projection** $\pi : V \to V$. **[3]**

(b) If $V = U \oplus W$ is a direct sum of subspaces $U, W \subseteq V$ then any element $v \in V$ can be written uniquely as $v = u + w$ with $u \in U$ and $w \in W$. Show that $\pi(v) = u$ defines a linear map $\pi : V \to V$. **[4]**

(c) Show for the map π constructed in (b) that $\mathrm{Im}(\pi) = U$ and $\mathrm{Ker}(\pi) = W$. **[6]**

(d) Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices over $\mathbb{R}$. Find a projection $\pi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ such that

$$\mathrm{Im}(\pi) = \langle I_n \rangle, \quad \mathrm{Ker}(\pi) = \{A \in M_n(\mathbb{R}) \mid \mathrm{Tr}(A) = 0\},$$

where I_n is the $n \times n$ identity matrix. Justify your answer. You may assume that the trace $\mathrm{Tr} : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear map. **[7]**

Question 4 [20 marks].

(a) Define the **characteristic polynomial** $p_A(x)$ and the **minimal polynomial** $m_A(x)$ of an $n \times n$ matrix A over a field $\mathbb{K}$. [3]

(b) Prove that if A is an $n \times n$ matrix and $B = P^{-1}AP$ for some invertible $n \times n$ matrix P then $p_B(x) = p_A(x)$. [5]

(c) Working over $\mathbb{R}$, determine $p_A(x)$ and $m_A(x)$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 2 & 1 & 2 \end{bmatrix}.$$

Show all your working. [8]

(d) Hence, or otherwise, show that the matrix A in part (c) **can not** be diagonalised over $\mathbb{R}$. You may refer in your proof to general results from Lectures. [4]

Question 5 [20 marks]. Let V be a vector space over a field $\mathbb{K}$ where 2 is invertible (i.e. not of characteristic 2) and $q : V \to \mathbb{K}$ a **quadratic form** on V.

(a) Give an example of a non-zero quadratic form over the field $\mathbb{F}_3$ of integers mod 3. [3]

(b) Explain how fixing a basis $v_1, \cdots, v_n$ of V associates to a quadratic form a corresponding function in n variables of the form $q(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$, for a certain symmetric matrix $A = (a_{ij})$. [5]

(c) Explain how **Sylvester's Law of Inertia** leads to non-negative integers s, t associated to a quadratic form over $\mathbb{R}$. Your explanation should include what it means for two symmetric matrices to be congruent and why this is relevant. [6]

(d) Find s, t for the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given with respect to its standard basis by

$$q(x, y, z) = -2xy + 2xz + 2yz + 2z^2.$$ [6]

(Hint: consider using identities such as $2xy = (x + y)^2 - x^2 - y^2$.)

End of Paper.