MTH5123 Differential Equations

Lecture Notes

Week 11

School of Mathematical Sciences
Queen Mary University of London

5 Stability of Solutions of ODEs

The subject of stability studies is to understand how a *change of initial conditions* or a *change in parameter values* of equations defining a dynamical system (e.g., coefficients in front of derivatives) affects the behaviour of the solutions, especially when the independent variable (usually interpreted as time t) tends to infinity, $t \to \infty$. The main goal is to establish criteria ensuring that the solution will change only slightly if a small (in an appropriate sense) change in the initial conditions or parameters is implemented. This type of question is of great importance for practical applications, as parameters of differential equations which govern real-life processes as, e.g., the functioning of mechanical aggregates or electronic devices, are known only approximately due to unpredictable changes in temperature, humidity or other properties of the environment. We will only discuss the stability of solutions of systems of two coupled first-order ODEs, but the basic ideas can be extended to any number of equations.
As usual we will use vector notation by writing down a system of two general non-autonomous ODEs in normal form as

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \; , \; \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \; , \; \mathbf{f} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix} \; . \tag{5.1}$$

Stability theory is mainly based on the following two definitions:

Definition: Lyapunov stability
A solution $\mathbf{y}_*(t)$ of (5.1) corresponding to the initial condition $\mathbf{y}_*(0) = \mathbf{a_1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ is called **Lyapunov stable** (or simply **stable**) if for any (arbitrarily small) $\epsilon > 0$ we can find a $\delta > 0$ such that if another initial condition $\mathbf{y}(0) = \mathbf{a_2} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is chosen inside a circle of radius δ around the initial point $\mathbf{y}_*(0)$ then for any time $t > 0$ the solution $\mathbf{y}(t)$ corresponding to the initial condition $\mathbf{y}(0)$

1. exists, and
2. will stay inside a "tube" of radius ϵ around the solution $\mathbf{y}_*(t)$, see Fig. (5.1).

In mathematical shortcut notation this definition reads

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \text{s.t.} \; \forall t > 0 \quad |\mathbf{y}(0) - \mathbf{y}_*(0)| < \delta \; \Rightarrow \; |\mathbf{y}(t) - \mathbf{y}_*(t)| < \epsilon \; .$$

Definition: asymptotic stability
The solution $\mathbf{y}_*(t)$ of (5.1) corresponding to the initial condition $\mathbf{y}_*(0) = \mathbf{a}$ is called **asymptotically stable** if it is

1. Lyapunov stable, and

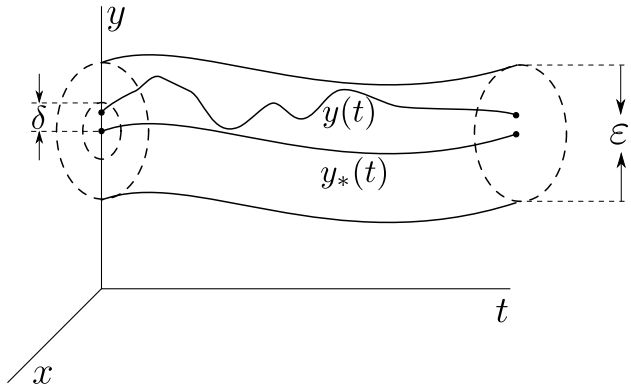

Figure 5.1: A sketch of the *stability tube* according to the definition of Lyapunov stability.
y

2. there exists a $\delta > 0$ such that the condition $|\mathbf{y}(0)-\mathbf{y}_*(0)| < \delta$ implies $|\mathbf{y}(t)-\mathbf{y}_*(t)| \to 0$ for $t \to \infty$.

y

Note:
1. Conditions 1. and 2. are *independent*, i.e., neither does 1. imply 2., nor does 2. imply 1. For the first direction see the counterexample below; the second direction is less obvious, but there are also counterexamples.
2. Making in (5.1) the change of variables $\mathbf{z}(t) \equiv \mathbf{y}(t)-\mathbf{y}_*(t)$ one can show that investigating the stability of any solution $\mathbf{y}_*(t)$ of the system (5.1) can always be reduced to investigating the stability of the *zero solution* $\mathbf{z}(t) = 0$ of the transformed system $\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z})$; see our previous discussion at the beginning of Section 4.1.1.

Reformulating the definition of (Lyapunov) stability in terms of $\mathbf{z}(t)$ we arrive at the following expression: Stability of the zero solution $\mathbf{z} = \mathbf{0}$, i.e., of the fixed point at $\mathbf{z} = \mathbf{0}$, means that for any $\epsilon > 0$ we can find a $\delta > 0$ such that $\forall t > 0$ $|\mathbf{z}(0)| < \delta$ implies $|\mathbf{z}(t)| < \epsilon$. The definition of asymptotic stability can be reformulated accordingly. From now on we will concentrate on stability of the zero solution only.

Example:
Is the zero solution $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the system $\dot{y}_1 = -4y_2$, $\dot{y}_2 = y_1$ (Lyapunov) stable? Is it asymptotically stable?

Solution:
The system can be written as $\dot{\mathbf{y}} = A\mathbf{y}$ with $A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$. The eigenvalues are purely imaginary, $\lambda_1 = 2i, \lambda_2 = -2i$, and the associated eigenvectors are $\mathbf{u}_{1,2} = \begin{pmatrix} 2 \\ \mp i \end{pmatrix}$. According to (4.19) the general solution of this system is given by

$$\mathbf{y}(t) = c_1 e^{2it} \begin{pmatrix} 2 \\ -i \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} 2 \\ i \end{pmatrix} .$$

Determining the two constants c_1, c_2 by imposing the initial conditions $y_1(0) = a, y_2(0) = b$ and expressing the solution in terms of real functions yields

$$y_1(t) = a\cos(2t) - 2b\sin(2t)\ ,\ y_2(t) = \frac{a}{2}\sin(2t) + b\cos(2t)\ .$$

As we have seen before, trajectories of this type are ellipses, $y_1^2 + 4y_2^2 = a^2 + 4b^2$. But this implies: Given any $\epsilon > 0$ let us choose $\delta = \epsilon/2$. Then by choosing the initial conditions (a, b) to be inside a circle of radius δ, that is, $a^2 + b^2 < \delta^2 = \epsilon^2/4$, we find that $\mathbf{y}^2(t) = y_1^2 + y_2^2 < y_1^2 + 4y_2^2 = a^2 + 4b^2 < 4(a^2 + b^2) < \epsilon^2$, hence $|\mathbf{y}| = \sqrt{y_1^2 + y_2^2} < \epsilon$ for any time t. We have thus shown that the zero solution $y_1 = y_2 = 0$ is Lyapunov stable. However, it is not asymptotically stable, as each solution rotates around its ellipse without approaching the origin for $t \to \infty$, see Fig. 5.

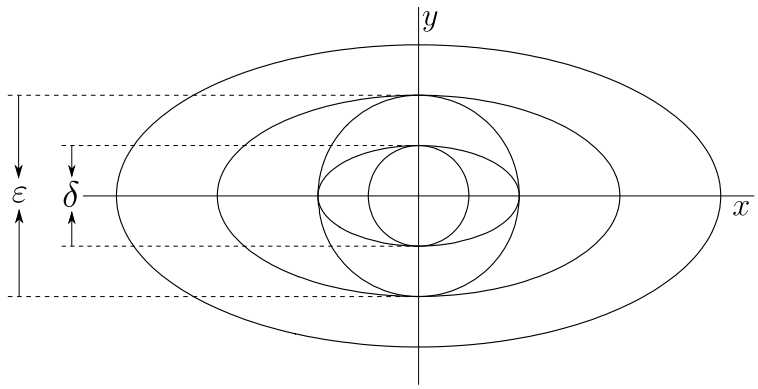

Figure 5.2: Sketch of the stability of the zero solution for elliptic trajectories providing an example that Lyapunov stability does not imply asymptotic stability.

5.1 Stability criteria for systems of two first-order linear ODEs with constant coefficents

Our goal is to formulate the stability conditions for the fixed point at $y_1 = y_2 = 0$ of any system of the form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\ , \tag{5.2}$$

where the matrix A is time independent. We will furthermore assume that A is characterized by distinct eigenvalues $\lambda_1 \neq \lambda_2$. One can then prove the following statement:

Theorem:
Define $s \equiv \max\{Re\lambda_1, Re\lambda_2\}$. Then the zero solution $\mathbf{y} = 0$ of (5.2) is

1. unstable for $s > 0$,
2. stable for $s = 0$, and
3. asymptotically stable for $s < 0$.

Instead of providing a proof we just outline the basic idea of it: If $s > 0$ then $e^{st} \to \infty$ for $t \to +\infty$. Hence at least one of the factors $|e^{\lambda_1 t}|, |e^{\lambda_2 t}|$ (or both) grows without bound in time, and the modulus of the solution must increase as well implying instability. Similarly, if $s < 0$ then $e^{st} \to 0$ for $t \to +\infty$ implying that *both* $|e^{\lambda_1 t}|, |e^{\lambda_2 t}| \to 0$ as $t \to \infty$. This means that the modulus of the solution must vanish asymptotically for $t \to \infty$ so that system is asymptotically stable. Finally, for $s = 0$ the eigenvalues are purely imaginary and complex conjugate. We know that in this case the trajectories are ellipses that neither approach zero nor go to infinity. Instead, they remain at a finite distance from the origin, hence this case is stable but not asymptotically stable. For a proof these ideas need to be formalized in terms of equations by starting from the general solution (4.19) for the general initial value problem $y_1(0) = a, y_2(0) = b$.

Note:
Although we considered only the case of distinct eigenvalues $\lambda_1 \neq \lambda_2$ the theorem can be generalized to $\lambda_1 = \lambda_2 = \lambda$ showing that also in this case the zero solution is unstable for $\lambda > 0$, stable for $\lambda = 0$ and asymptotically stable for $\lambda < 0$.

5.2 Lyapunov function method for investigating stability

Consider again the general system of two first order ODEs written in normal form (5.1). Suppose that $\mathbf{y}(t)$ is a solution of (5.1). Then for any continuously differentiable function $V(\mathbf{y})$ defined on the same domain as $\mathbf{y}(t)$ one can define its values at any moment of time t *on the solution* $\mathbf{y}(t)$ as $v(t) \equiv V(y_1(t), y_2(t))$. We will need the expression for the time derivative $\dot{v} = \frac{dv}{dt}$ of such a function, which by using the chain rule of differentiation can be obtained to

$$\dot{v} = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 \,. \tag{5.3}$$

Using (5.1) in the form of $\dot{y}_1 = f_1(t, y_1, y_2), \dot{y}_2 = f_2(t, y_1, y_2)$ we obtain

$$\dot{v} = \frac{\partial V}{\partial y_1} f_1(t, y_1, y_2) + \frac{\partial V}{\partial y_2} f_2(t, y_1, y_2) \equiv \mathcal{D}_f(V) \,, \tag{5.4}$$

where we introduced the notation $\mathcal{D}_f(V)$. In Calculus II you have learned that this equation defines the *directional derivative of* V *along* $\mathbf{f}$. Within our specific context, the above expression is called the **orbital derivative**. Note that $\mathcal{D}_f(V)$ is determined for any value of $\mathbf{y}$ solely by the functional form of $V(\mathbf{y})$ and the form of the right-hand side of the system (5.1) without the need to know the explicit solution of the latter system.
$\mathcal{D}_f(V)$ can be used to formulate the following statement, which we give without proof:

Theorem: Lyapunov Stability Theorem
Let $\mathbf{y}(t) = 0$ be a solution of (5.1) and assume that inside the circle $0 < |\mathbf{y}| < R$ there exists a continuously differentiable function $V(\mathbf{y})$ satisfying

1. $V(\mathbf{y} = 0) = 0$
2. $V(\mathbf{y} \neq 0) > 0$
3. The derivative of V along $\mathbf{f}$ is non-positive, $\mathcal{D}_f(V) \leq 0$ for $(y_1, y_2) \neq (0, 0)$.

Then the zero solution $\mathbf{y}(t) = 0$ is **stable**.

The function $V(\mathbf{y})$ featuring in this theorem is called the **Lyapunov function** of the system (5.1). While such a function can be found for certain classes of differential equations, see the following example, unfortunately there does not exist a systematic way of how to construct it for a given system of differential equations.

Note:
If the third condition is replaced by $\mathcal{D}_f(V) < 0$ being *strictly negative* one can prove that the zero solution $\mathbf{y}(t) = 0$ is **asymptotically stable**, which is called the **Lyapunov Asymptotic Stability Theorem**.

Example:
Verify that the function $V(y_1, y_2) = y_1^2 + y_2^2$ is a valid Lyapunov function for the system

$$\dot{y}_1 = -y_2 - y_1^3 \,,\; \dot{y}_2 = y_1 - y_2^3 \,.$$

Is the zero solution asymptotically stable?

Solution:
$V(y_1, y_2)$ satisfies the first and the second condition in the Lyapunov Stability Theorem. For any $(y_1, y_2) \neq (0, 0)$ we have

$$\mathcal{D}_f(V) = \frac{\partial V}{\partial x}\dot{y}_1 + \frac{\partial V}{\partial y}\dot{y}_2 = 2y_1\left(-y_2 - y_1^3\right) + 2y_2(y_1 - y_2^3) = -2(y_1^4 + y_2^4) < 0$$

so that the zero solution is not only stable but even asymptotically stable.

Theorem: Consider a continuously differentiable function $V(\mathbf{y})$ satisfying

1. $V(\mathbf{y} = 0) = 0$
2. $V(\mathbf{y} \neq 0) > 0$

If the system (5.1) is autonomous and can be written as

$$\dot{y}_1 = -\frac{\partial V}{\partial y_1} \tag{5.5}$$

$$\dot{y}_2 = -\frac{\partial V}{\partial y_2} \tag{5.6}$$

then the dynamical system is called a **gradient flow**, $\mathbf{y} = \mathbf{0}$ is an equilibrium solution of the dynamical system which is Lyapunov stable. The function $V(\mathbf{y})$ is a Lyapunov function of the dynamical system called **potential**.

Proof:
In order to show that $\mathbf{y} = \mathbf{0}$ is an equilibrium solution we note that if condition 1. and 2. are satisfied then $\mathbf{y} = \mathbf{0}$ is a minimum of $V(\mathbf{y})$, thus

$$\left.\frac{\partial V}{\partial y_1}\right|_{\mathbf{y}=\mathbf{0}} = 0, \quad \left.\frac{\partial V}{\partial y_2}\right|_{\mathbf{y}=\mathbf{0}} = 0. \tag{5.7}$$

Hence $\mathbf{y} = \mathbf{0}$ is an equilibrium solution of the gradient flow.

The function $V(\mathbf{y})$ is a Lyapunov function for the gradient flow.Indeed it satisfies conditions 1. and 2. and its the orbital derivative is non-negative, as

$$D_f(V) = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 = -\left[\left(\frac{\partial V}{\partial y_1}\right)^2 + \left(\frac{\partial V}{\partial y_2}\right)^2\right] \leq 0. \tag{5.8}$$

From the Lyapunov stability theorem it follows that the equilibrium solution $\mathbf{y} = \mathbf{0}$ is Lyapunov stable.

Example: Verify that the following dynamical system is a gradient flow and determine its Lyapunov function (potential):

$$\dot{y}_1 = -y_1 - y_1 y_2^2, \quad \dot{y}_2 = -2y_2 - y_1^2 y_2 \tag{5.9}$$

We want to express the dynamical system as a gradient flow of $V(y_1, y_2)$, i.e.

$$f_1(y_1, y_2) = -y_1 - y_1 y_2^2 = -\frac{\partial V}{\partial y_1} \tag{5.10}$$

$$f_2(y_1, y_2) = -2y_2 - y_1^2 y_2 = -\frac{\partial V}{\partial y_2}. \tag{5.11}$$

Integrating the first equation we get

$$V(y_1, y_2) = -\int f_1(y_1, y_2) dy_1 + g(y_2) = \frac{1}{2}y_1^2 + \frac{1}{2}y_1^2 y_2^2 + g(y_2). \tag{5.12}$$

Imposing $f_2(y_1, y_2) = -2y_2 - y_1^2 y_2 = -\frac{\partial V}{\partial y_2}$ we obtain $g'(y_2) = -2y_2$ leading to $g(y_2) = y_2^2 + C$. Imposing $V(\mathbf{0}) = 0$ we get

$$V(y_1, y_2) = \frac{1}{2}y_1^2 + y_2^2 + \frac{1}{2}y_1^2 y_2^2 \tag{5.13}$$

It is easy to check that $V(\mathbf{y}) > 0$ for $\mathbf{y} \neq \mathbf{0}$. Therefore $V(y_1, y_2)$ conditions 1. and 2. and is the Lyapunov function (potential function) of the considered dynamical system. Moreover, the considered dynamical system is the gradient flow of $V(\mathbf{y})$ and its equilibrium solution $\mathbf{y} = \mathbf{0}$ is Lyapunov stable.

The Lyapunov function method enables to investigate the stability of whole classes of systems of ODEs. For example, along these lines one can prove the following important generalization of the theorem on p.4, which we state without proof:

Theorem:
Let us consider a nonlinear system of two ODEs of the form

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \text{higher order nonlinear terms}\,, \tag{5.14}$$

where the matrix A is time independent and characterized by the two eigenvalues λ_1, λ_2. Then

1. if both $Re\lambda_1 < 0$ and $Re\lambda_2 < 0$ then the zero solution of (5.14) is asymptotically stable.
2. If at least one of $Re\lambda_1, Re\lambda_2$ is positive then the zero solution of (5.14) is unstable.
3. If $max\{Re\lambda_1, Re\lambda_2\} = 0$ then the stability of the zero solution is determined not only by A but also by the properties of the nonlinear terms, i.e., the zero solution may be stable for some nonlinear terms but unstable for others.

Note:
The third case implies that **linear stability analysis does not work** if $max\{Re\lambda_1, Re\lambda_2\} = 0$.

Example:
Determine the maximal range of the values of the parameter a for which the zero solution of the system

$$\dot{y_1} = y_1 + (2-a)y_2, \quad \dot{y_2} = ay_1 - 3y_2 + (a^2 - 2a - 3)y_1^2$$

is (**i**) unstable, (**ii**) stable.

Solution:
The linear part of the system is obtained by simply discarding the nonlinear terms in the second equation, as can be verified by Taylor expansion. Hence it is described by the matrix $A = \begin{pmatrix} 1 & 2-a \\ a & -3 \end{pmatrix}$ whose characteristic equation is $\lambda^2 + 2\lambda + (a^2 - 2a - 3) = 0$. The two roots are given by

$$\lambda_1 = -1 + \sqrt{-a^2 + 2a + 4}\,, \; \lambda_2 = -1 - \sqrt{-a^2 + 2a + 4}\,.$$

If the two roots are complex conjugate we have $Re\{\lambda_{1,2}\} = -1 < 0$, hence the zero solution is asymptotically stable. We conclude that an instability may occur only for values of a where both roots are real and $\lambda_1 > 0$. This implies $\sqrt{-a^2 + 2a + 4} > 1$ so that $-a^2 + 2a + 4 > 1$ or equivalently

$$a^2 - 2a - 3 = (a+1)(a-3) < 0\,,$$

hence $-1 < a < 3$. Thus for $a \in (-1, 3)$ the zero solution is unstable. Correspondingly, for $a < -1$ or $a > 3$ the zero solution must be asymptotically stable. For $a = -1$ or $a = 3$ we have $\lambda_1 = 0$ while $\lambda_2 = -2 < 0$. Therefore in this case the situation depends on the nonlinear terms. But precisely for these parameter values of a the nonlinear term in our original system vanishes, due to $a^2 - 2a - 3 = 0$. The system thus becomes linear with eigenvalues $\lambda_1 = 0, \lambda_2 = -2$, hence by our theorem on page 4 of this document for $a = -1, 3$ the zero solution is stable but not asymptotically stable.