Differential Equations
Lesson 1 Week 1

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Lectures and Tutorials

- Schedule:
- Lesson 1-2 Wednesday 11:00-13:00 Live Arts Two LT
- Lesson 3 Thursday 17:00-18:00 Live Arts Two LT

You will be assigned to one of the following three tutorials
(check your timetable to know which applies)

- Tutorial 1: Thursday 10:00-11:00 PP2
- Tutorial 2: Thursday 12:00-13:00 PP2
- Tutorial 3: Thursday 16:00-17:00 Graduate Ctr:GC201

Lecture notes

- Lecture note material available on QMPlus:
    - Typed in lecture notes
    - Handwritten lecture notes

Reading list

- Main reference: Types lecture notes
- Other references:
- J. C. Robinson: An introduction to Ordinary Differential Equations (Cambridge University Press)
- Available from the library! (See link in QM+)

Formative Assessment

- Each week, at the tutorial we will cover
- The formative assignment and the mock quiz for the week, available on the QMPlus page

Assessed courseworks

- You will have two assessed courseworks (quizzes).
- Every assessed coursework is worth 10% of the final marks
- You will have one week to complete assessed courseworks but once you open the quiz you need to complete it within 48 hours
- The 2 assessed courseworks will be posted in weeks 6,11

Final exam

- The final exam will account for 80% of the final mark

Feedback

Feedback on your assessed courseworks:

Personalised feedback

Quiz questions: You will be able to see your scores soon after the submission deadline

General feedback

General feedback will be given during the tutorial

where we will go over the most challenging questions of the assessed

courseworks and the common mistakes.

Support Learning Hour

Thursday 2:00-3:00pm MB521

Online Forum

For any question on the module material that you would like to ask there are two ways to received feedback and answers:

- You can ask the question during the live lectures and tutorials
- You can post the question on the online forum

Participation to the online forum is highly beneficial as it increases the interactions between fellow students.

- The online forum will be monitored twice a week

Outline of the Lesson

- ODEs versus PDEs
- Dependent and independent variables
- ODEs and their order
- What is a solution of a ODE?
- The simplest linear ODE
- Examples of applications

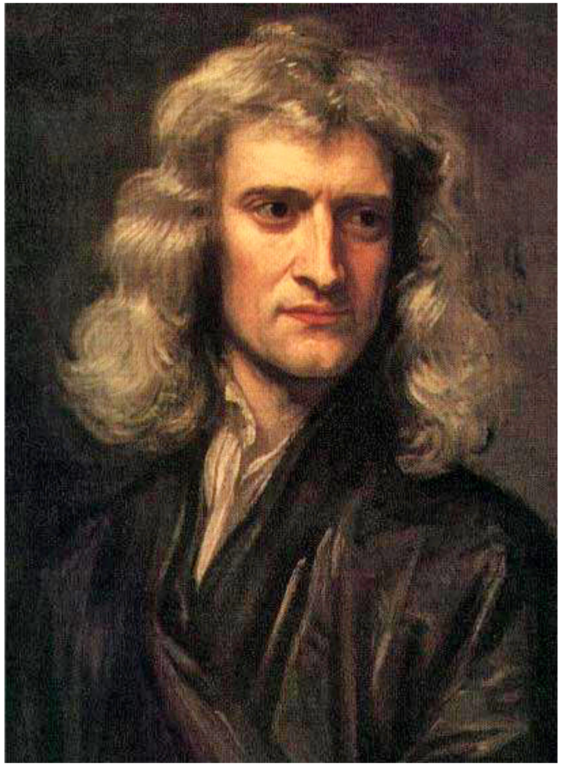
Isaac Newton
(25 December 1642 – 20 March 1726)

Driven by the need to understand

the physical word

including gravitation and mechanics

Isaac Newton founded Calculus



Newton
groundbreaking role
in mathematics and physics

Newton groundbreaking role in mathematics and physics can be summarised by the following two points:

- He wrote the differential equations describing the motion of masses in presence of forces (such as gravitation) valid in classical mechanics

- He founded Calculus for solving the differential equations that he formulated

In this module we will expand your analytical skills
to solve differential equations and to
solve problems coming from several applications

Ordinary differential equations (ODEs)

Ordinary differential equations are equations

satisfied by a function of a single variable

involving the function, its derivatives and its argument

Example: $$\frac{dy}{dx} = x^2$$

This equation is solved when we find all the functions $y(x)$ that satisfy it

x indicates the independent variable

y indicates the dependent variable.

Partial differential equations (PDEs)

Ordinary differential equations differs from partial differential equations

Partial differential equations are satisfied by functions of two or more variables and involve the function together with its partial derivatives and its argument

Example: $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ (wave equation)

This equation is solved when we find all the functions $u(x, t)$ that satisfy it

x, t indicate the two independent variables

u indicates the dependent variable.

In this module we will cover only ODEs

The notation

The independent variable and the dependent variable

can be also indicated by other letters:

Function	Independent variable	Dependent variable
$y(x)$	x	y
$y(t)$	t	y
$x(t)$	t	x

Derivative notation

To indicate the derivative we can use the following notation

Derivative with respect to x

$$\frac{dy}{dx} \longleftrightarrow y'(x)$$

$$\frac{d^2y}{dx^2} \longleftrightarrow y''(x)$$

$$\vdots \qquad \vdots$$

$$\frac{d^ny}{dx^n} \longleftrightarrow y^{(n)}(x)$$

Derivative with respect to t

$$\frac{dy}{dt} \longleftrightarrow \dot{y}(t)$$

$$\frac{d^2y}{dt^2} \longleftrightarrow \ddot{y}(t)$$

$$\vdots \qquad \vdots$$

$$\frac{d^ny}{dt^n} \longleftrightarrow y^{(n)}(t)$$

Ordinary differential equations

An ordinary differential equation of order n for a function $y(x)$

is an equation of the form

$$\mathscr{F}\left(y^{(n)}(x), y^{(n-1)}(x), \ldots, y''(x), y'(x), y, x\right) = 0$$

(where the $n-th$ derivative $y^{(n)}(x)$ occurs in $\mathscr{F}$)

Order of a ODE

The order of an ODE

is the order to the highest derivative present in the equation

Examples:

$y' \sin x - y^2 + x^3 = 0$	$\mathscr{F}(y', y, x) = 0$	**1st order ODE**
$(y')^2 - y''' + e^x = 0$	$\mathscr{F}(y''', y', x) = 0$	**3rd-order ODE**
$e^x y'' + \sin y = 0$	$\mathscr{F}(y'', y, x) = 0$	**2nd-order ODE**

Normal form of an ODE

The normal form of an n-order ODE is an explicit expression of the n-order derivative of the type

$$y^{(n)}(x) = f(y^{(n-1)}, y^{(n-2)} \dots, y'', y', y, x)$$

- **Examples**: 1st-order ODE in normal form $y' = f(x, y)$

  2nd-order ODE in normal form $y'' = f(x, y, y')$

Solution of an ODE

A solution of of a given ODE is a function for which the ODE becomes and identity

Example: The function $y = \bar{y}(x)$ is a solution of the ODE

$$\frac{dy}{dx} = f(x, y)$$

if

$$\frac{d\bar{y}(x)}{dx} = f(x, \bar{y}(x))$$

is an identity

You can use other notation for

the independent variable and the dependent variables!

Differential equations and variables

ODE	Independent variable	Dependent Variable	Normal form?	Order
$\frac{dy}{dt} = yt$	t	y	Yes	1
$\ddot{y} = -g$	t	y	Yes	2
$\ddot{x} = -\frac{k}{m}(x - x_0)$	t	x	Yes	2
$e^{-x}y' = y^2$	x	y	No	1

The simplest linear ODE

The simplest (linear) ODE you already know how to solve from Calculus is

$$\frac{dy}{dx} = f(x)$$

which has as solution the anti-derivative of $f(x)$

$$y(x) = F(x) = \int f(x)dx + C$$

Indeed we have the identity

$$\frac{dF(x)}{dx} = f(x)$$

Please refresh your Calculus I and Algebra

by

answering the

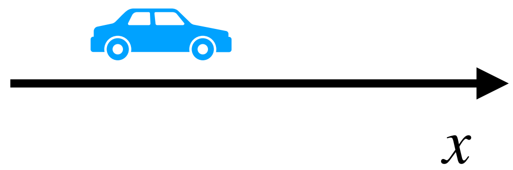
Revision questions of pre-request knowledge from previous modules

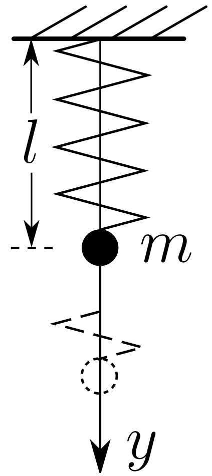
posted in the QM+ page under Week 1

Physics

Velocity and acceleration of one dimensional motion

Function	Velocity	Acceleration
$x(t)$	$\dfrac{dx}{dt}, \dot{x}$	$\dfrac{d^2x}{dt^2} = \dfrac{d\dot{x}}{dt}, \ddot{x}$
$y(t)$	$\dfrac{dy}{dt}, \dot{y}$	$\dfrac{d^2y}{dt^2} = \dfrac{d\dot{y}}{dt}, \ddot{y}$





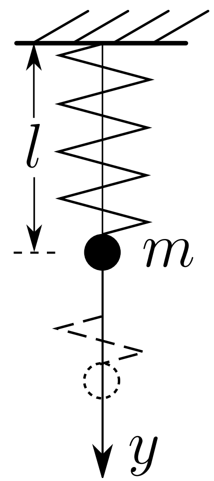
Newton Law

To solve problem in physics (classical mechanics) Newton formulated the three Newton Laws

For a mass hanging from the ceiling with a spring the Newton 2nd Law

is a ODE that reads

$$m\ddot{y} = f(\dot{y}, y, t)$$



where $f(\dot{y}, y, t)$ expresses all the forces acting on the mass:

the gravitational force, the elastic force (due to the spring) and friction

Classical mechanics: the birth of differential equations

At time t=0 you launch a mass vertically from a given height h at a given velocity v_0.

- Which is the height of the mass at time t?
- When does this mass reach the floor?
- At which velocity does it reach the floor?

$$m\ddot{y} = -mg$$

$$y(0) = h$$
$$\dot{y}(0) = v_0$$

Financial mathematics: Continuously Varying Interest Rates

Suppose you put x in your bank account at time $t = 0$, and the interest is continuously compounded with a rate $r(t)$ that changes in times.

- What is the amount $D(t)$ that you will have in your account at time t?

$$\dot{D}(t) = D(t)r(t)$$

$$D(0) = x$$

Modelling epidemic spreading

You want to model a pandemic

- What is the dynamics for a given R_0?
- How many would be the removed individuals (immunised or deceased) at time t?
- The SIR dynamics involving a given fraction of susceptible S, infected I and removed individuals R

$$S + I + R = 1$$

$$\dot{I} = R_0 I(1 - I - R) - I$$
$$\dot{R} = I$$

$$I(0) = I_0$$
$$R(0) = R_0$$

Climate change equation

- **Energy Balance models**

- Expresses how the temperature T of the Earth changes in time due to the balance between the radiation absorbed by the Sun S and the radiation emitted by the Earth $e\sigma T^4$, where the radiation emitted by Earth depends on the presence of Green House Gases in the atmosphere that change the emissivity e of the Earth.

$$c\rho\frac{dT}{dt} = (1-a)S - e\sigma T^4$$

This equation can be used to inform us in the implementation of the 2016 Paris Agreement to keep the Earth's temperature increase below 1.5 Degrees Centigrade

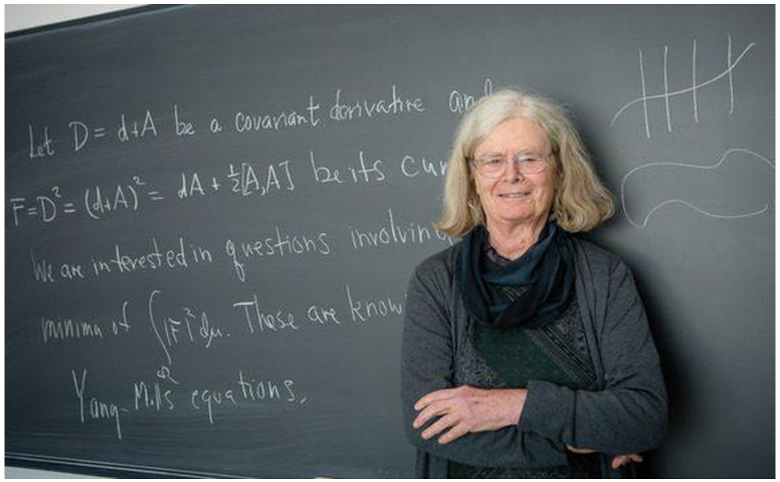
Differential equations are useful to solve

many applied problems!

Notable mathematicians in differential equations

Katherine Johnson
(1918-2020)

Miryam Mirzakhani
(1977-2017)
Field Medal 2014

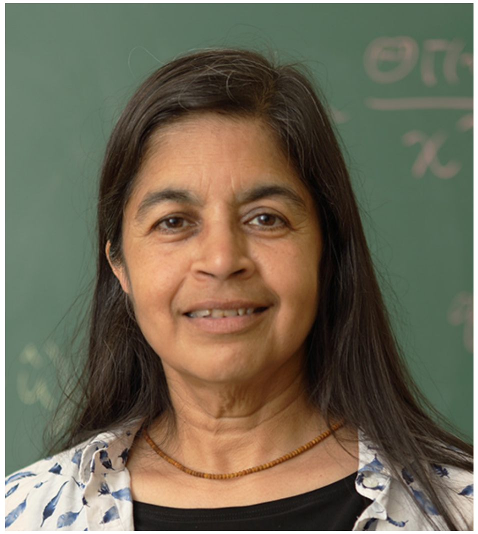


Karen Keskulla Uhlenbeck (born 1942) Abel Prize 2019

Notable scientists working with Differential equational and Dynamical Systems and applications

Syukuro Manabe Princeton University

Artur Avila Cordeiro de Melo University of Zurich



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