Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH9000 / MTH9000P: Sample Module

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: F. Examiner, S. Examiner

This is optional. Here, one should put any special instructions (if any) relating to the exam. Otherwise, do not use this functionality.
For example: In this exam $0 \in \mathbb{N}$. Also, all expressions should be simplified as much as possible.

Question 1 [15 marks].

(a) Give a useful parametrisation of all solutions in positive integers x, y, z to the equation $x^2 + y^2 = z^2$. [**5**]

(b) Prove that the equation $x^4 + y^4 = z^2$ has no solutions in which x, y and z are positive integers. [**5**]

(c) Classify all quadruples (x, y, z, n) of positive integers such that $n \geq 3$ and the equation $x^n + y^n = z^n$ holds. Justify your answer. [**5**]

Question 2 [25 marks].

(a) Define what it means for a function to be **analytic** on some open subset of $\mathbb{C}$. [**5**]

Let s be a complex number such that $\Re(s) > 1$. Define $\zeta(s)$ to be the number:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

(In all cases, n^{-s} means $\exp(-s \log n)$, with $\log n \in \mathbb{R}$.)

(b) State how to define the analytic continuation of ζ to all of $\mathbb{C} \setminus \{1\}$. [**10**]

(c) Show that all non-real zeros of this analytic continuation have real part equal to $\frac{1}{2}$. [**10**]

Question 3 [20 marks].

(a) We consider packings in two dimensions.

(i) Prove that no packing of unit circles in the Euclidean plane has density exceeding $\frac{\pi}{2\sqrt{3}}$. [**5**]

(ii) What is the maximum possible density for packing ellipses with semi-minor axis 1 and semi-major axis a into Euclidean space? Justify your answer. [**5**]

(b) (i) Prove **without the assistance of a computer** that no packing of unit spheres in $\mathbb{R}^3$ has density exceeding $\frac{\pi}{3\sqrt{2}}$. [**5**]

(ii) Generalise your result to packing ellipsoids whose semi-axes are a, b and c. [**5**]

Question 4 [10 marks]. Prove that any even integer at least 8 can be written as the sum of two distinct (positive) primes.

Question 5 [30 marks]. State and prove the Classification of Finite Simple Groups. *[You must clearly establish the existence of all groups concerned, especially if they are defined by having involution centralisers of a certain type.]* [**30**]

End of Paper – An appendix of 2 pages follows.

A (hyper-)volume formula

The formula for the (hyper-)volume of an n-dimensional ball of radius r is

$$\frac{(\pi r^2)^{n/2}}{\Gamma(\frac{n}{2}+1)},$$

where, for $x > 0$, $\Gamma(x)$ is defined to be the real integral $\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$.

Some trigonometric identities

$$\begin{aligned}
\sin^2 A + \cos^2 A &= 1\\
\tan^2 A + 1 &= \sec^2 A\\
1 + \cot^2 A &= \operatorname{cosec}^2 A\\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Continue to next page

Some derivatives

In the table below, a, b and c are constants.

$f(x)$	$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x))$
ax^b	abx^{b-1}
$cf(x)$	$cf'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(g(x))$	$g'(x)f'(g(x))$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\log x$	$\frac{1}{x}$

End of Appendix.