Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH9000 / MTH9000P: Sample Module

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: F. Examiner, S. Examiner

This is optional. Here, one should put any special instructions (if any) relating to the exam. Otherwise, do not use this functionality.
For example: In this exam $0 \in \mathbb{N}$. Also, all expressions should be simplified as much as possible.

Question 1 [15 marks].

(a) Give a useful parametrisation of all solutions in positive integers x, y, z to the equation $x^2 + y^2 = z^2$. **[5]**

(b) Prove that the equation $x^4 + y^4 = z^2$ has no solutions in which x, y and z are positive integers. **[5]**

(c) Classify all quadruples (x, y, z, n) of positive integers such that $n \geq 3$ and the equation $x^n + y^n = z^n$ holds. Justify your answer. **[5]**

Question 2 [25 marks].

(a) Define what it means for a function to be **analytic** on some open subset of $\mathbb{C}$. **[5]**

Let s be a complex number such that $\Re(s) > 1$. Define $\zeta(s)$ to be the number:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

(In all cases, n^{-s} means $\exp(-s \log n)$, with $\log n \in \mathbb{R}$.)

(b) State how to define the analytic continuation of ζ to all of $\mathbb{C} \setminus \{1\}$. **[10]**

(c) Show that all non-real zeros of this analytic continuation have real part equal to $\frac{1}{2}$. **[10]**

Question 3 [20 marks].

(a) We consider packings in two dimensions.

(i) Prove that no packing of unit circles in the Euclidean plane has density exceeding $\frac{\pi}{2\sqrt{3}}$. [**5**]

(ii) What is the maximum possible density for packing ellipses with semi-minor axis 1 and semi-major axis a into Euclidean space? Justify your answer. [**5**]

(b) (i) Prove **without the assistance of a computer** that no packing of unit spheres in $\mathbb{R}^3$ has density exceeding $\frac{\pi}{3\sqrt{2}}$. [**5**]

(ii) Generalise your result to packing ellipsoids whose semi-axes are a, b and c. [**5**]

Question 4 [10 marks]. Prove that any even integer at least 8 can be written as the sum of two distinct (positive) primes.

Question 5 [30 marks]. State and prove the Classification of Finite Simple Groups. *[You must clearly establish the existence of all groups concerned, especially if they are defined by having involution centralisers of a certain type.]* [**30**]

End of Paper – An appendix of 2 pages follows.

A (hyper-)volume formula

The formula for the (hyper-)volume of an n-dimensional ball of radius r is

$$\frac{(\pi r^2)^{n/2}}{\Gamma(\frac{n}{2}+1)},$$

where, for $x > 0$, $\Gamma(x)$ is defined to be the real integral $\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$.

Some trigonometric identities

$$\begin{aligned}
\sin^2 A + \cos^2 A &= 1\\
\tan^2 A + 1 &= \sec^2 A\\
1 + \cot^2 A &= \operatorname{cosec}^2 A\\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Some derivatives

In the table below, a, b and c are constants.

$f(x)$	$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)$
ax^b	abx^{b-1}
$cf(x)$	$cf'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(g(x))$	$g'(x)f'(g(x))$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\log x$	$\frac{1}{x}$

End of Appendix.