Pre-Sessional Math and Statistics

Lecture 1: Simple functions and the basics of present value

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Aim of the Pre-Sessional Math & Statistics Course

• Review key mathematical and statistical concepts and tools

• Show examples of how these tools are used in Economics and Finance

• Ensure a solid foundation for your study in the MSc program
Structure of the Course

• Six lectures in the series
• Each lecture lasts about 90 minutes, recorded in 9-10 segments
• Online quiz offers opportunity for self-tests
• There is no exam at the end
• Lecturer: Dr. Yu Zheng
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Reference Books

For Mathematics

For Statistics

For Econometrics

Most available as e-books in the Library
Lecture 1: Simple Functions and the Basics of Present Value

• Definition of functions
• Commonly used functions and their applications in economics and finance:
  • Linear and quadratic functions
  • Exponential and logarithmic functions
• Definition of sequences and series
• Present value calculations

Functions

Definition. Given two sets $X$ and $Y$, a function from $X$ to $Y$ is a rule that associates with each number of $X$, one and only one number of $Y$.

Examples of functions: say $X = Y = \mathbb{R} \equiv \{\text{the set of all real numbers}\}$.

- Take any number in $X$ and multiply it by 10
- Take any number in $X$ and square it
- Take any number in $X$ and add 1 to it
Functions

Definition. Given two sets $X$ and $Y$, a function from $X$ to $Y$ is a rule that associates with each number of $X$, one and only one number of $Y$.

The set $X$ is called the domain of the function and the set of numbers in $Y$ associated with the numbers in $X$ is called the range of the function. Denote the rule of association by $f$, we can write the function as

$$f : X \rightarrow Y$$

or as

$$y = f(x), \quad x \in X.$$  

$y$ is often referred to as the dependent variable and $x$ as the independent variable. We also say $y$ is the value of the function $f$ at $x$. 

Functions

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Examples of functions again: $X = Y = \mathbb{R} \equiv \{\text{the set of all real numbers}\}$

- Take any number in $X$ and multiply it by 10: $f(x) = 10x, x \in \mathbb{R}$
- Take any number in $X$ and square it: $g(x) = x^2, x \in \mathbb{R}$
- Take any number in $X$ and add 1 to it: $h(x) = x + 1, x \in \mathbb{R}$
Functions

Examples of functions again: \( X = Y = \mathbb{R} \equiv \{ \text{the set of all real numbers} \} \).

- Take any number in \( X \) and multiply it by 10: \( f(x) = 10x, x \in \mathbb{R} \).
- Take any number in \( X \) and square it: \( g(x) = x^2, x \in \mathbb{R} \).
- Take any number in \( X \) and add 1 to it: \( h(x) = x + 1, x \in \mathbb{R} \).

What is the domain of \( f(x) \)?

What is the range of \( g(x) \)?

What is the value of the function \( f \) at 2?

What is the value of the function \( g \) at -1?

What is the value of the function \( h \) at 100?
Functions

What is the graph of a function?

Example: How to draw $f(x) = 2x + 1, x \in \mathbb{R}$?

What really happens is that we want to draw a set of ordered pairs $\{(x, y) | y = 2x + 1\}$ in the coordinate system x-y

- Draw a few points, (-1,-1), (0,1), (2,5), in the x-y plane
- Connect the points
Functions

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What really happens is that we want to draw a set of ordered pairs \( \{(x, y) \mid y = 2x + 1\} \) in the coordinate system x-y

• Draw a few points, (-1,-1), (0,1), (2,5), in the x-y plane
• Connect the points.

Refresh your memories of the coordinate system:
Simple Functions: Linear Functions

A **linear** function takes the form

\[ y = ax + b, \quad x \in \mathbb{R}. \]

The *parameter* \(a\) is called the **slope** and the *parameter* \(b\) is called the **intercept term** of the linear function.

Note the difference between a parameter and a variable.

Examples: \(y = 2 + 3x\) and \(y = 2 - 3x\). What are the slope and intercept of each of these functions?
Simple Functions: Linear Functions

Example: $y = 2 + 3x$
Simple Functions: Linear Functions

Example: $y = 2 - 3x$
Simple Functions: Linear Functions

\[ y = ax + b, \quad x \in \mathbb{R} \]

• For \( a > 0 \), an upward sloping line
• For \( a < 0 \), a downward sloping line
• For \( a = 0 \), a horizontal line

• For \( b > 0 \), the line intercepts the y-axis above the origin
• For \( b < 0 \), the line intercepts the y-axis below the origin
• For \( b = 0 \), the line goes through the origin.
Simple Functions: Quadratic Functions

A quadratic function takes the form
\[ y = ax^2 + bx + c, \quad x \in \mathbb{R}, a \neq 0. \]

If the parameter \( a = 0 \), the function becomes a linear function with slope \( b \) and intercept \( c \).

If the parameter \( a > 0 \), the graph of the function is U-shaped.

If the parameter \( a < 0 \), the graph of the function is inverted U-shaped.

Examples: \( y = x^2 + 4x + 3 \) and \( y = -x^2 + 6x - 5 \). Which is U-shaped and which is inverted U-shaped?
Simple Functions: Quadratic Functions

Example: \( y = x^2 + 4x + 3 \)
Simple Functions: Quadratic Functions

Example: $y = -x^2 + 6x - 5$
Simple Functions: Quadratic Functions

\[ y = ax^2 + bx + c, \quad x \in \mathbb{R}, a \neq 0 \]

Where does the graph intercepts with the x-axis?

Equivalently, what are the roots of (or solutions to) the quadratic equation

\[ ax^2 + bx + c = 0 \]
Simple Functions: Quadratic Functions

A technique called “completing the square”:

\[
\begin{align*}
  a \left( x^2 + \frac{b}{a}x \right) + c &= a \left( x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 \right) - a \left( \frac{b}{2a} \right)^2 + c \\
  &= a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a} = 0 \\
  \Rightarrow \left( x + \frac{b}{2a} \right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
  \Rightarrow x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
  \Rightarrow x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.
\end{align*}
\]
Simple Functions: Exponential Functions

An **exponential** function takes the form

\[ y = a \cdot b^x, \quad x \in \mathbb{R}. \]

The parameter \( b \) is called the **base** of the function. Suppose \( a > 0 \).

If the parameter \( b > 1 \), the function increases in \( x \).

If the parameter \( 0 < b < 1 \), the function decreases in \( x \).

Examples: \( y = 5^x \) and \( y = 0.6^x \). Which is increasing and which is decreasing?
Simple Functions: Exponential Functions

Example: $y = 5^x$
Simple Functions: Exponential Functions

Example: \( y = 0.6^x \)
Simple Functions: Exponential Functions

Exponential functions are commonly used to express exponential growth such as interest compounding and depreciation. More on this later.

• You have just deposited £100 in your savings account this month. The monthly interest rate is 0.1%. Assuming no withdrawal, express the month-end balance of your account as a function of month $t$:
  $$B(t) = 100 \times 1.001^t, \quad t = 1,2,3, \ldots$$

• You have just bought a fax machine valued at £500. The value of the machine depreciates at 20% per year. Express the year-end value of the machine as a function of year $t$:
  $$D(t) = 500 \times 0.8^t, \quad t = 1,2,3, \ldots$$
Simple Functions: Logarithmic Functions

A logarithmic function takes the form
\[ y = \log_b x, \quad x \in \mathbb{R}_+ \equiv \{a \text{ set of all positive numbers}\}, b > 0, b \neq 1. \]

If \( b = e \approx 2.718 \ldots \), then we have the natural logarithm, write \( y = \ln x \).
If \( b = 10 \), then we have the common logarithm, write \( y = \log x \).

A log function is the inverse of the exponential function:
\[ y = b^x \Rightarrow x = \log_b y. \]

If the base \( b > 1 \), the function increases in \( x \).
If the base \( 0 < b < 1 \), the function decreases in \( x \).

Examples: \( y = \log_3 x \) and \( y = \log_{0.5} x \)
Simple Functions: Logarithmic Functions

Example: $y = \log_3 x$
Simple Functions: Logarithmic Functions

Example: \( y = \log_{0.5} x \).
Simple Functions: Logarithmic Functions

Logarithmic Functions are commonly used to transform economic or financial variables thanks to the following properties:

\[
\log a^b = b \log a \\
\log_a (b \times c) = \log_a b + \log_a c
\]

- Take log of the Gross Domestic Product to spot changes in growth rates
- Take log of asset returns
Simple Functions: Logarithmic Functions

Example. Country A’s GDP growth rate averages at 2% per year from 1950 to 1980 and accelerates to 10% per year from 1981 to 2010.

Figure A: GDP Level (Normalized to 1 in 1950)

Figure B: log GDP (Normalized to 0 in 1950)
Simple Functions: Logarithmic Functions

Example. The stock price of company ABC varies over time as shown in the following table:

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>100</td>
<td>120</td>
<td>180</td>
</tr>
</tbody>
</table>

The log return from time 1 to 2: \(\log(120/100) = 7.92\%\)
The log return from time 2 to 3: \(\log(180/120) = 17.62\%\)
The log return from time 1 to 3: \(\log(180/100) = 25.53\%\)

Note: \(7.92\% + 17.62\% = 25.53\%\)! Time additive
Simple returns don’t have this property: \(20\% + 50\% \neq 80\%.\)
Sequences, Series, and Present-Value Calculations

Definition. A **sequence** is a function whose domain is the positive integers.

Examples.

- \( f(n) = 2n, \ n = 1, 2, 3, 4 \ldots \) or \( 2, 4, 6, 8, 10, \ldots \)
- \( f(n) = \frac{1}{n}, \ n = 1, 2, 3, 4 \ldots \) or \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \)

It is sometimes convenient to denote a sequence by \( a_n = f(n) \)

An important application of sequences in finance is the determination of the **present value** of a sum of money to be received at some time in the future.
Suppose you invest $X$ pounds today at an annual rate of return $r$. Then at the end of the first year, your investment will generate

$$V = X(1 + r)$$

Then we say the present value of amount $V$ to be received in one year’s time is

$$PV = \frac{V}{1 + r} = X$$

Example. Suppose you will receive £120 at the end of one year and the annual interest rate is 20%. What is the present value of that £120?

$$PV = \frac{\text{£120}}{1 + 20\%} = \text{£100}$$
Sequences, Series, and Present-Value Calculations

More generally, the present value $PV_t$ of amount $V$ to be received $t$ periods from now when the interest rate is $r$ per period and compounding occurs at the end of each period is

$$PV_t = \frac{V}{(1 + r)^t}, \quad t = 1, 2, 3, \ldots$$

- This is a sequence
- As $t$ increases, $PV_t$ decreases. The longer you have to wait, the more you discount future benefit and the lower the PV of the benefit
- We refer to $\frac{1}{1+r}$ as the discount factor
Sequences, Series, and Present-Value Calculations

• What is a period? How frequently are interests compounded within a period?

Suppose a period is a year and the annual interest is 12%. You invest £1000 in the account for two years. Compute the value of your investment after two years in each of the following scenarios.

**Scenario A:** Annual compounding.

\[ V = £1000 \left(1 + \frac{12\%}{1}\right)^2 = £1254.40 \]

**Scenario B:** Semiannual compounding.

\[ V = £1000 \left(1 + \frac{6\%}{2}\right)^4 = £1262.48 \]

**Scenario C:** Monthly compounding.

\[ V = £1000 \left(1 + \frac{1\%}{12}\right)^{24} = £1269.73 \]
Sequences, Series, and Present-Value Calculations

If we compound interest $n$ times a year and the annual interest rate is $r$, then the value of £$P$ invested for $t$ years is

$$V = P \left(1 + \frac{r}{n}\right)^{nt}$$

If we compound infinitely many times a year, or let $n \to +\infty$, we have the case of **continuous compounding**, and the value of £$P$ invested at interest rate $r$ for $t$ periods is worth

$$V = Pe^{rt}$$

at the end of $t$ periods.

**Scenario D:** Continuous compounding.

$$V = £1000e^{12\% \times 2} = £1271.25$$
Sequences, Series, and Present-Value Calculations

The present value of £$V$ to be received $t$ years from now is given by the following formulae:

- If the interest rate is $r$ per year and compounding $n$ times per year
  \[
  PV_t = \frac{V}{(1 + r/n)^{nt}}
  \]

- If the interest rate is $r$ per year under continuous compounding
  \[
  PV_t = V e^{-rt}.
  \]
Sequences, Series, and Present-Value Calculations

**Definition.** If \( a_t, \ t = 1, 2, 3, \ldots \) is a sequence, then \( s_n = \sum_{t=1}^{n} a_t, n = 1, 2, 3, \ldots \), is called a **series**.

A **geometric series** takes the form

\[
s_n = \sum_{t=1}^{n} a \rho^{t-1} = a + a \rho + a \rho^2 + \cdots + a \rho^{n-1} = a \left( \frac{1 - \rho^n}{1 - \rho} \right).
\]

Moreover, for \( 0 < \rho < 1 \), we have

\[
\lim_{n \to +\infty} s_n = \frac{a}{1 - \rho}
\]
Sequences, Series, and Present-Value Calculations

The present value of a stream of payments of £V at the end of each year for T years, with the annual interest rate r and annual compounding, is

$$PV_T = \frac{V}{1 + r} + \frac{V}{(1 + r)^2} + \cdots + \frac{V}{(1 + r)^T}$$

$$= \sum_{t=1}^{T} \frac{V}{(1 + r)^t} = \frac{V}{1 + r} \left[ 1 + \frac{1}{1 + r} + \left(\frac{1}{1 + r}\right)^2 + \cdots + \left(\frac{1}{1 + r}\right)^{T-1} \right]$$

$$= \frac{V}{1 + r} \left[ \frac{1 - \left(\frac{1}{1 + r}\right)^T}{1 - \frac{1}{1 + r}} \right]$$
Sequences, Series, and Present-Value Calculations

If we can receive the stream of payments until forever, or $T \rightarrow +\infty$, the present value becomes

$$PV_\infty = \frac{V}{r}.$$ 

$$\lim_{T \rightarrow +\infty} \frac{V}{1 + r} \left[ \frac{1 - \left(\frac{1}{1 + r}\right)^T}{1 - \frac{1}{1 + r}} \right] = \frac{V}{1 + r} \left[ \frac{1}{1 - \frac{1}{1 + r}} \right] = \frac{V}{r}.$$
Sequences, Series, and Present-Value Calculations

Example (project evaluation). An investment project requires an immediate costs of £200,000. It generates net revenues of £15,000 per year starting at the end of the first year and continuing forever. The annual interest rate is 9%.

The present value of the benefits of the project is

\[ PV = \frac{15,000}{0.09} = 166,666.67 \]

The net present value of the project is the present value of benefits minus the present value of costs:

\[ NPV = 166,666.67 - 200,000 < 0 \]
Sequences, Series, and Present-Value Calculations

How do we decide whether a project is worth taking up or not?

• If the NPV of a project is negative, then the project is not worthwhile
• If the NPV of a project is positive, then the project is worth investing in
Sequences, Series, and Present-Value Calculations

Example (project evaluation) cont’d. What if the interest is 7.5% instead?

The NPV of the project is then:

\[
NPV = \frac{15,000}{0.075} - 200,000 = 0.
\]

Break even!

The *internal rate of return* of this project is 7.5%
Sequences, Series, and Present-Value Calculations

Definition. The **internal rate of return** of a project or investment is the rate of interest that equates the present value of benefits and costs.

How do we decide whether a project is worth taking up or not?

- If the market interest rate is higher than the internal rate of return, then the project is not worthwhile.
- If the market interest rate is lower than the internal rate of return, then the project is worth investing in.
Sequences, Series, and Present-Value Calculations

Example (project evaluation) cont’d. An investment project requires an immediate costs of £200,000. It generates net revenues of £15,000 per year starting at the end of the first year and continuing forever. The annual market interest rate is 9%. What is the internal interest rate and is this project worth investing in?

The internal rate of return is

\[
\frac{15,000}{r_{\text{internal}}} = 200,000 \Rightarrow r_{\text{internal}} = 7.5\%.
\]

Since it is lower than the market return of 9%, this project is not worthwhile.