$$C^T = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -5 \\ -9 & 2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & -9 \\ -3 & 1 & 2 \\ 4 & -5 & 0 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 29 & -21 & -24 \\ -21 & 27 & -7 \\ -24 & -7 & 85 \end{pmatrix} \qquad 4+9+16$$

symmetric

$$\frac{1}{2}(C+C^T) = \frac{1}{2}\begin{pmatrix} 4 & -2 & -5 \\ -2 & 2 & -3 \\ -5 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -\frac{5}{2} \\ -1 & 1 & -\frac{3}{2} \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{pmatrix}$$

(b) Let A be a square matrix.
Prove that $A^T A$ and $\frac{1}{2}(A+A^T)$ are symmetric.

Proof

By definition of symmetric matrix I need to prove that

- $$A^T A = (A^T A)^T$$

$$\frac{1}{2}(A+A^T) = \left(\frac{1}{2}(A+A^T)\right)^T$$

D is symmetric iff $(d_{ij}) = (d_{ji})$

$$D = D^T$$

(1) $$(EF)^T = F^T E^T$$

(2) $$(E+F)^T = E^T + F^T$$

Making use of the property (1) seen in class we have that

- $$(A^T A)^T = A^T (A^T)^T = A^T A$$

Making use of (2) we have that

$$\left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A+A^T)^T =$$

$$= \frac{1}{2}\left(A^T + (A^T)^T\right)$$

$$= \frac{1}{2}(A^T + A) = \frac{1}{2}(A+A^T) \quad \square$$

A B

$m \times n$ AB $p \times q$

$n = p$

Question 3 [25 marks].

(a) Let

$$C = \begin{pmatrix} 2 & 1 & -9 \\ -3 & 1 & 2 \\ 4 & -5 & 0 \end{pmatrix}.$$

Evaluate C^T, $C^T C$, $\frac{1}{2}(C + C^T)$. **[4]**

(b) Prove that for any square matrix A, the matrices $A^T A$ and $\frac{1}{2}(A + A^T)$ are both symmetric. **[6]**

(c) If we take $B = \frac{1}{2}(A + A^T)$, then prove $(A - B)^T = B - A$. **[5]**

(d) Are the matrices $A^T A$ and AA^T always equal? Either prove this result or state a counter-example. **[4]**

(e) Prove that if A is invertible, then so is $A^T A$. **[6]**

We have seen in class that if A is invertible then $(A^T)^{-1} = (A^{-1})^T$

$$(A^T A)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T \quad \square$$

If A and B are invertible $(AB)^{-1} = B^{-1}A^{-1}$

(c) Let $B = \frac{1}{2}(A + A^T)$.

Then, $A - B = A - \frac{1}{2}(A + A^T)$

By the properties of the multiplication by scalars we have

$$(A-B) = A - \frac{1}{2}(A + A^T) = \left(A - \frac{1}{2}A\right) - \frac{1}{2}A^T$$

$$= \frac{1}{2}A - \frac{1}{2}A^T = \frac{1}{2}(A - A^T)$$

$$(A-B)^T = \frac{1}{2}(A^T - A)$$

$$B - A = \frac{1}{2}(A + A^T) - A = \frac{1}{2}A + \frac{1}{2}A^T - A = \frac{1}{2}A^T - \frac{1}{2}A$$

$$= \frac{1}{2}(A^T - A) \quad \square$$

(d) Are A^TA and AA^T equal?

$A = A^T$ symmetric $\qquad A^TA = A^2$

$\qquad AA^T = A^2$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$