Today

- Final material on game theory (examinable)

Revision

Week 4 - extreme point solutions
basic feasible solutions

Week 10/11 Game theory

General 2-player games

Example 12.1. Suppose that Rosemary and Colin are working on a joint project. Each of them can choose to "work hard" or "goof off." Both of them must work hard together to receive a high mark for the project. Both have utility 3 for receiving a high mark utility 1 for goofing off (regardless of what mark they receive) and utility 0 for working hard but not receiving a high mark. Give the payoff matrix for this game.

Colin

Rosemary	work hard (w)	goof off (g)
work hard (w)	(3, 3)	(0, 1)
goof off (g)	(1, 0)	(1, 1)

not zero-sum game

Rosemary's set of strategies $R = \{w, g\}$

Colin's set of strategies $C = \{w, g\}$

Write out Rosemary's payoff function $u_1 : R \times C \to \mathbb{R}$

Write out Colin's payoff function $u_2 : R \times C \to \mathbb{R}$

Does our example have any Nash equilibria?

Rosemary		Colin: work hard (w)	Colin: goof off (g)
	work hard (w)	(3, 3)	(0, 1)
	goof off (g)	(1, 0)	(1, 1)

How can we systematically and quickly find all pure Nash equilibrium.

Example 12.2. Find all pure Nash equilibria for the games with the following payoff matrices.

	c_1	c_2	c_3
r_1	$(0,-1)$	$(1,2)$	$(2,-1)$
r_2	$(2,1)$	$(0,-1)$	$(2,1)$
r_3	$(0,2)$	$(-1,1)$	$(1,-1)$

	c_1	c_2	c_3
r_1	$(1,0)$	$(0,1)$	$(1,-1)$
r_2	$(-1,1)$	$(1,0)$	$(0,1)$

Recap quiz (paraphrased defns/theorems)

Consider an LP in standard equation form

$$\begin{aligned} &\text{maximize} && \underline{c}^T \underline{x} \\ &\text{subject to} && A\underline{x} = \underline{b}, \quad \underline{x} \geq 0. \end{aligned}$$

An extreme point solution is a ________ solution $\underline{x}$ such that $\underline{x}$ cannot be written as ____________ where $\underline{y}, \underline{z}$ are distinct ______ ______ and $\lambda \in$ ______.

A basic feasible solution is a ________ solution $\underline{x}$ in which the ________ entries of $\underline{x}$ correspond to __________ columns of A.

Last time we proved two results

1. Every LP (in standard equation form) has an ________ solution that is an ________________ solution (provided it has at least one _________ solution).

2. Given an LP in standard equation form every ___________ solution is an ____________________ solution and vice versa.

(proof not completed)

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(b) Consider the following linear program in standard equation form:

$$\begin{array}{ll} \text{maximise} & x_1 + 2x_2 - 3x_3 + 7x_5 \\ \text{subject to} & \begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 3, \\ x_1 + 2x_2 + 7x_3 + x_5 &= 3, \\ 2x_1 + 4x_2 + 7x_3 + x_6 &= 6, \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned} \end{array}$$

For each of the following values of $\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3, x_4, x_5, x_6)$ say whether or not this value is a **basic feasible solution** of this linear program and also whether or not it is an **extreme point solution** of this linear program. Justify your answers. [**9**]

(i) $\mathbf{x}^\mathsf{T} = (1, 1, 0, 0, 0, 0)$

(ii) $\mathbf{x}^\mathsf{T} = (1, 0, 0, 2, 2, 4)$

(iii) $\mathbf{x}^\mathsf{T} = (0, 0, 0, 3, 3, 6)$

(c) Consider an arbitrary linear program in standard equation form:

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Suppose that $\mathbf{x}$ is an optimal solution to this linear program. Show that if $\mathbf{x}$ is **not** an extreme point solution then we can express $\mathbf{x}$ as $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ where $\lambda \in (0,1)$ and $\mathbf{y}$ and $\mathbf{z}$ are two different optimal solutions of this program. [**8**]

Basic terminology

strategy, outcome, payoff matrix, zero-sum game, mixed strategy

What is a pure Nash equilibrium in words/symbols?

What is the security level of a strategy in words/symbols?

What is best security level for a player?

How are they related?

How do we compute expected payoff
when players use mixed strategy

What is a mixed Nash equilibrium in words/symbols?
What is the security level of a mixed strategy
in words/symbols

How are they related.
How do we write LP's to find optimal mixed strategy?
i.e. mixed strategy with best security.

Question 5 [24 marks].

(a) In the following question, let $\beta \in \mathbb{R}$ be a fixed constant. Suppose a zero-sum 2-player game has the following payoff matrix, given from the perspective of the row player:

	1	2
1	β	6
2	-6	0

(i) Suppose that $\beta = 0$. Give the security levels for each of the row and column players' strategies. List all pure Nash equilibria for this game or explain why the game does not have a pure Nash equilibrium. [**4**]

(ii) For what range of possible values for β is $(1, 2)$ a pure Nash equilibrium for this game? Justify your answer. [**6**]

(iii) For what range of possible values for β does this game have a general Nash equilibrium? Justify your answer. [**4**]

(b) Consider the following 2-player game. Rosemary and Colin each select a number n from the set $\{1, 2, 3\}$. If they choose the same number, neither player wins anything. Otherwise, if the sum of their numbers is at least 5, both of them win £1. Finally, if their numbers do not match and do not sum to at least 5, then the player who selected the largest number n wins £n and the other player loses £n.

(i) Give the payoff matrix for this game (as usual, suppose that Rosemary is the row player and give her payoff first in each cell). [**4**]

(ii) Is this a zero sum game? Justify your answer. [**2**]

(iii) List all pure Nash equilibria for this game. [**4**]

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(b) Consider the 2-player zero-sum game with the following payoff matrix (which is given, as usual, from the perspective of the row player).

	c_1	c_2.
r_1	6	-6
r_2	3	9

(i) Write a linear program that finds the optimal mixed strategy for the row player (i.e. the mixed strategy with the best security level). You do not have to solve this linear program. **[6]**

(ii) Consider the mixed strategy $\mathbf{x}$ for the row player and $\mathbf{y}$ for the column player given by $\mathbf{x}^\mathsf{T} = (1/3, 2/3)$ and $\mathbf{y}^\mathsf{T} = (5/6, 1/6)$. Show that this pair of strategies is a mixed Nash equilibrium for this game. **[8]**