Queen Mary
University of London

Main Examination period 2020 – May – Semester B

MTH4115 / MTH4215: Vectors & Matrices SOLUTIONS

Duration: 2 hours

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Examiners: O. Jenkinson, R. Johnson

Question 1 [20 marks]. Let A, B, C be points in 3-space with respective position vectors $\mathbf{a} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}-1\\3\\3\end{pmatrix}$, $\mathbf{c} = \begin{pmatrix}1\\-3\\3\end{pmatrix}$. Determine:

(a) The length of the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$; **[3]**

(b) A unit vector in the direction of $\mathbf{a}$; **[3]**

(c) $\mathbf{a}\cdot\mathbf{b}$; **[3]**

(d) $\mathbf{a}\times\mathbf{b}$; **[3]**

(e) A vector equation for the line through A and B; **[4]**

(f) A Cartesian equation for the plane containing A, B and C. **[4]**

Solutions [All parts are routine calculations]:

(a) $\mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix}1\\0\\7\end{pmatrix}$, which has length $\sqrt{1+0+49} = \sqrt{50} = 5\sqrt{2}$.

(b) $\mathbf{a}$ has length $\sqrt{2}$, so the unit vector in the direction of $\mathbf{a}$ is $\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$.

(c) $\mathbf{a}\cdot\mathbf{b} = -1+0+3 = 2$.

(d) $\mathbf{a}\times\mathbf{b} = \begin{pmatrix}-3\\-4\\3\end{pmatrix}$.

(e) $\mathbf{r} = \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$, $\lambda\in\mathbb{R}$, is such an equation, which in this case becomes
$\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-2\\3\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(f) $\mathbf{n} = (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{pmatrix}-2\\3\\2\end{pmatrix} \times \begin{pmatrix}0\\-3\\2\end{pmatrix} = \begin{pmatrix}12\\4\\6\end{pmatrix}$ is orthogonal to this plane, and since A is contained in the plane then an equation for it is $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n} = 18$, which in Cartesian form is $12x+4y+6z = 18$, or alternatively $6x+2y+3z = 9$.

Question 2 [20 marks].

(a) What is the definition of a **bound vector**? [2]

(b) What is the definition of a **free vector**? [2]

(c) What does it mean to say that a bound vector **represents** a free vector? [2]

(d) What is the definition of a **parallelogram**? [2]

(e) State the **parallelogram axiom**. [2]

(f) Given free vectors $\mathbf{u}$ and $\mathbf{v}$, explain in terms of a parallelogram how their **sum** $\mathbf{u}+\mathbf{v}$ is defined. [3]

(g) Let O be a fixed origin in 3-space, let P and Q be any points in this space, and let R be the point such that the figure $OPQR$ is a parallelogram. Let $\mathbf{p}$ and $\mathbf{q}$ denote the position vectors for P and Q, respectively. Find an expression for the free vector represented by $\overrightarrow{RP}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. [4]

(h) With the points as in (g) above, let S be the point such that the figure $ORPS$ is a parallelogram. Find an expression for the free vector represented by $\overrightarrow{SQ}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. [3]

Solutions [Parts (a) to (f) are bookwork. Part (g) appeared on an exercise sheet. Part (h) is unseen.]:

(a) A **bound vector** is a directed line segment in 3-space; in other words, a bound vector is determined by its starting point, its length, and its direction (provided the length is non-zero).

(b) A **free vector** is determined by its length and its direction (provided that the length is not 0).

(c) We say that a bound vector **represents** a free vector if it has the same length and direction as the free vector.

(d) The figure $ABCD$ is a **parallelogram** if $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent the same free vector.

(e) If $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent the same vector, then $\overrightarrow{BC}$ and $\overrightarrow{AD}$ represent the same vector.

(f) Given vectors $\mathbf{u}$ and $\mathbf{v}$ we define the sum $\mathbf{u}+\mathbf{v}$ as follows. Pick any point A and let B,C,D be points such that $\overrightarrow{AB}$ represents $\mathbf{u}$, $\overrightarrow{AD}$ represents $\mathbf{v}$ and $ABCD$ is a parallelogram. Then $\mathbf{u}+\mathbf{v}$ is the vector represented by $\overrightarrow{AC}$.

(g) The free vector represented by $\overrightarrow{RP}$ is $2\mathbf{p}-\mathbf{q}$.

(h) The free vector represented by $\overrightarrow{SQ}$ is $2\mathbf{q}-2\mathbf{p}$.

Question 3 [20 marks]. Let Π be the plane with equation $2x+y+z=1$, let l be the line with equations $x=y=z$, and let Q be the point with position vector $\mathbf{q}=\begin{pmatrix}0\\1\\2\end{pmatrix}$.

(a) Determine the distance between the point Q and the plane Π. **[4]**

(b) Determine the coordinates of the point on Π that is closest to Q. **[4]**

(c) Determine the distance between the point Q and the line l. **[4]**

(d) Determine the point of intersection of the line l and the plane Π. **[4]**

(e) If l' is the line in the direction orthogonal to Π and passing through Q, then determine the distance between l and l'. **[4]**

Solutions: [All parts are fairly routine use of formulae from lectures and practiced on exercise sheets; part (e) should be slightly more challenging]

(a) The vector $\mathbf{n}=\begin{pmatrix}2\\1\\1\end{pmatrix}$ is orthogonal to Π, so this distance (using the formula derived in lectures) is $|\mathbf{q}\cdot\mathbf{n}-1|/|\mathbf{n}|=2/\sqrt{6}=\sqrt{6}/3$.

(b) Using the formula from lectures, this closest point has position vector

$$\mathbf{q}-\left(\frac{\mathbf{q}\cdot\mathbf{n}-1}{|\mathbf{n}|^2}\right)\mathbf{n}=\begin{pmatrix}0\\1\\2\end{pmatrix}-(1/3)\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-2/3\\2/3\\5/3\end{pmatrix},$$

so its coordinates are $(-2/3,2/3,5/3)$.

(c) The line l has vector equation $\mathbf{r}=\lambda\mathbf{u}$ where $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, so the required distance is $|\mathbf{u}\times\mathbf{q}|/|\mathbf{u}|$ (a formula from lectures). Now $\mathbf{u}\times\mathbf{q}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, so $|\mathbf{u}\times\mathbf{q}|=\sqrt{6}$, and $|u|=\sqrt{3}$, therefore the required distance is $|\mathbf{u}\times\mathbf{q}|/|\mathbf{u}|=\sqrt{6}/\sqrt{3}=\sqrt{2}$.

(d) The point of intersection has coordinates $(1/4,1/4,1/4)$.

(e) The line l' has vector equation $\mathbf{r}=\mathbf{q}+\lambda\mathbf{n}$, so by a formula from lectures the required distance is $|\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})|/|\mathbf{u}\times\mathbf{n}|$. Now $\mathbf{u}\times\mathbf{n}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$, so $|\mathbf{u}\times\mathbf{n}|=\sqrt{2}$, and $\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})=1-2=-1$, so the required distance is $|\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})|/|\mathbf{u}\times\mathbf{n}|=1/\sqrt{2}$.

Question 4 [20 marks].

(a) What are the three types of **elementary row operation** that can be performed on a matrix? **[3]**

(b) Describe in detail the **Gaussian elimination algorithm** for putting a matrix in row echelon form. **[4]**

(c) Consider the linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 2x_1 & + & 3x_2 & - & 2x_3 & + & 7x_4 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 1\,. \end{array}$$

(i) Write down the augmented matrix of the system. **[3]**

(ii) Bring the augmented matrix to row echelon form. Indicate which elementary row operation you use at each step. **[6]**

(iii) Identify the leading and free variables of the reduced system, and write down the solution set of the system. **[4]**

Solutions [Parts (a) and (b) are bookwork. Part (c) is similar to examples seen in lectures and on exercise sheets]:

(a) The three types of operation are as follows:

Type I: interchanging two rows;
Type II: multiplying a row by a non-zero scalar;
Type III: adding a multiple of one row to another row.

(b) The algorithm is as follows:

Step 1: If the matrix consists entirely of zeros, stop — it is already in row echelon form.

Step 2: Otherwise, find the first column from the left containing a non-zero entry (call it a), and move the row containing that entry to the top position.

Step 3: Now multiply that row by $1/a$ to create a leading 1.

Step 4: By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row. All further operations are carried out on the other rows.

Step 5: Repeat steps 1-4 on the matrix consisting of the remaining rows

The process stops when either no rows remain at Step 5 or the remaining rows consist of zeros.

(c) (i) The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 4 & 1 \\ 2 & 3 & 4 & 1 & 0 \\ 2 & 3 & -2 & 7 & 2 \\ 2 & 3 & 4 & -2 & 1 \end{array}\right).$$

(ii) Using elementary row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - R_1$ gives

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 4 & 1 \\ 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & -3 & 3 & 1 \\ 0 & 0 & 3 & -6 & 0 \end{array}\right),$$

then using elementary row operations $R_3 \to R_3 + R_2$ and $R_4 \to R_4 - R_2$ gives

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 4 & 1 \\ 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{array}\right),$$

then using elementary row operation $R_3 \leftrightarrow R_4$ gives

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 4 & 1 \\ 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

then using elementary row operations $R_1 \to (1/2)R_1$, $R_2 \to (1/3)R_2$ and $R_3 \to (-1/3)R_3$ gives

$$\left(\begin{array}{cccc|c} 1 & 3/2 & 1/2 & 2 & 1/2 \\ 0 & 0 & 1 & -1 & -1/3 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which is in row echelon form.

(iii) The leading variables are x_1, x_3 and x_4, while the free variable is x_2. We see that $x_4 = -1/3$, and $x_3 = x_4 - 1/3 = -2/3$, and $x_1 = -(3/2)x_2 - (1/2)x_3 - 2x_4 + 1/2 = -(3/2)x_2 + 3/2$, so the solution set is

$$\left\{ \left(-\frac{3}{2}\alpha + \frac{3}{2}, \alpha, -\frac{2}{3}, -\frac{1}{3}\right) : \alpha \in \mathbb{R} \right\}.$$

Question 5 [20 marks]. Let

$$A = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -3 \\ 1 & 2 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 3 & -2 \\ -3 & 1 \\ 2 & -1 \end{pmatrix} \text{ and } C = \begin{pmatrix} -2 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 1 & -2 \end{pmatrix}.$$

(a) For each of the expressions B^2, BA, CB, CA, $A-4B$, and $-A+5C$, state whether or not it exists, and if it exists then evaluate it. **[12]**

(b) Is the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ invertible? Justify your answer. **[2]**

(c) Find a 2×2 matrix D such that $D \neq 0_{2\times 2}$ but $D^2 = 0_{2\times 2}$. **[3]**

(d) Find 2×2 matrices E and F such that $EF = 0_{2\times 2}$ but $FE \neq 0_{2\times 2}$. **[3]**

Solutions [Part (a) is routine, part (b) is fairly routine, parts (c) and (d) appear on an exercise sheet]:

(a) B^2 does not exist.

BA does not exist.

$$CB = \begin{pmatrix} 6 & -1 \\ 3 & -3 \\ -10 & 5 \end{pmatrix}.$$

$$CA = \begin{pmatrix} 1 & 8 & 0 \\ 1 & -2 & 3 \\ -1 & -3 & -6 \end{pmatrix}.$$

$A - 4B$ does not exist

$$-A+5C = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 3 \\ -1 & -2 & 0 \end{pmatrix} + \begin{pmatrix} -10 & -10 & 15 \\ 10 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix} = \begin{pmatrix} -10 & -9 & 12 \\ 9 & 5 & 3 \\ -6 & 3 & -10 \end{pmatrix}.$$

(b) The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is **not** invertible. To see this, either note that its determinant is zero, or alternatively note that for any other 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 2c & 2d \end{pmatrix},$$

which cannot equal the identity matrix.

(c) One possibility is $D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(d) One possibility is to take $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $EF = 0_{2\times 2}$, but $FE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E \neq 0_{2\times 2}$.

End of Paper.