Queen Mary
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Main Examination period Sample Paper – May/June – Semester B

MTH4115 / MTH4215: Vectors and Matrices

Examiners: C. Garetto, W. Huang, M. Lewis

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Question 1 [25 marks].

(a) Let $A = (-1, 3, -2)$, $B = (0, 1, 5)$ and $C = (-2, 1, 7)$. Compute **[10]**

$$\frac{|\overrightarrow{BA}|}{|\overrightarrow{AC}|^2}\,.$$

(b) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, and P be the point in $\mathbb{R}^3$ with position vector $\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$.

(i) Write the parametric equations of the line l through P in the direction of the vector $\mathbf{u}$. **[5]**

(ii) Does the point $Q = (1, 2, 1)$ lie on the line l? Justify your answer with a short argument. **[5]**

(c) Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and let R be the point in $\mathbb{R}^3$ with coordinates (a, b, c). Prove that $|\mathbf{v}|$ is the length of the segment OR. **[5]**

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Solutions:

(a) **Similar to examples seen in lecture notes.**

Let $A = (-1, 3, -2)$, $B = (0, 1, 5)$ and $C = (-2, 1, 7)$. By direct computations we have

$$\overrightarrow{BA} = \begin{pmatrix} -1 \\ 2 \\ -7 \end{pmatrix}$$

and

$$|\overrightarrow{BA}| = \sqrt{1 + 4 + 49} = \sqrt{54}.$$

Since,

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ -2 \\ 9 \end{pmatrix}$$

we have that

$$|\overrightarrow{AC}|^2 = 1 + 4 + 81 = 86.$$

Concluding,

$$\frac{|\overrightarrow{BA}|}{|\overrightarrow{AC}|^2} = \frac{\sqrt{54}}{86} = \frac{3\sqrt{6}}{86}.$$

[**10**]

(b) (i) **Definition seen in lecture notes.**

The parametric equations of the line l through P in the direction of the vector $\mathbf{u}$ are given by

$$x = -1 + \lambda,$$
$$y = 2,$$
$$z = 0 + 3\lambda,$$

with $\lambda \in \mathbb{R}$. [**5**]

(ii) **Similar to examples seen in lecture notes.**

The point $Q = (1, 2, 1)$ belongs to the line l if there exists $\lambda \in \mathbb{R}$ such that

$$1 = -1 + \lambda,$$
$$2 = 2,$$
$$1 = 3\lambda.$$

From the first equation we get $\lambda = 2$, however $\lambda = 2$ does not fulfil the third equation. This means that the system is inconsistent and therefore Q does not belong to l. [**5**]

(c) **Proof available in the lecture notes.** [**5**]

We assume that $R \neq O$ otherwise the statement is trivial. To compute the length of the segment OR we project R on the xy-plane. We get the point $S = (a, b, 0)$.

The length of the segment OS by Pythagoras' theorem on the xy-plane is given by $\sqrt{a^2 + b^2}$. Let us consider the triangle OSR with sides OS and QR. By applying Pythagoras' Theorem again, we have that the length of OR is given by

$$\sqrt{a^2 + b^2 + c^2} = |\mathbf{v}| \ .$$

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Question 2 [25 marks].

In a three dimensional space $\mathbb{R}^3$, consider plane Π_1 given by the Cartesian equation $x + y + z = 6$, plane Π_2 given by the Cartesian equation $x + 2y + 3z = 14$, and plane Π_3 given by the Cartesian equation $x + 3y + 2z = 13$.

(a) Write down the linear system A, whose solutions are the intersection of these three planes. Write down the associated homogeneous system B to this linear system A. [**5**]

(b) Bring the augmented matrix of the homogeneous system B obtained in (a) to row echelon form. State the leading and free variables of the system in this form, and find all solutions of B. [**10**]

(c) Based on the solutions of B, state how many points are in the intersection of the three planes. Write down all solutions of the linear system A. [**10**]

Solutions:

(a) **Application of definitions seen in lecture notes.**

The linear system A, whose solutions are the intersection of the three planes is

$$\begin{aligned} x + y + z &= 6, \\ x + 2y + 3z &= 14, \\ x + 3y + 2z &= 13. \end{aligned}$$

The associated homogeneous system B is,

$$\begin{aligned} x + y + z &= 0, \\ x + 2y + 3z &= 0, \\ x + 3y + 2z &= 0. \end{aligned}$$

(b) **Similar to examples seen in lecture notes.**

The augmented matrix of system B is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right).$$

Performing the Gaussian Algorithm on the augmented matrix above gives its row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right) \xrightarrow[R3-R1]{R_2-R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_3-2R_2}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

In the row echelon form, we have 3 non-zero rows and thus 3 leading 1s, which correspond to the 3 leading variables, x, y and z. There are no free variables. Thus, the system has a unique solution, which is its trivial solution $(0, 0, 0)$.

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(c) **Similar to examples seen in lecture notes.**

By part (b), the system B has three leading variables, and the trivial solution is therefore its only solution. According to Theorem 6.3.6 in the lecture notes, an $n \times n$ system is consistent and has a unique solution if and only if the only solution of the associated homogeneous system is the zero solution. Thus, system A has a unique solution. This means the three planes intersect at a single point.

The same sequence of elementary row operations (given by the Gaussian Algorithm) can bring the augmented matrix of a system and its associated homogeneous system to the row echelon form. Thus, we can bring the system A to row echelon form by:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 3 & 2 & 13 \end{array}\right) \xrightarrow[R3-R1]{R_2-R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 1 & 7 \end{array}\right) \xrightarrow{R_3-2R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

The solution of system A is therefore

$$\begin{aligned} z &= 3 , \\ y &= 8 - 2z = 2 , \\ x &= 6 - z - y = 1 . \end{aligned}$$

Question 3 [25 marks].

(a) Let

$$C = \begin{pmatrix} 2 & 1 & -9 \\ -3 & 1 & 2 \\ 5 & -4 & 0 \end{pmatrix} .$$

Evaluate C^T, $C^T C$, $\frac{1}{2}(C + C^T)$. [**4**]

(b) Prove that for any square matrix A, the matrices $A^T A$ and $\frac{1}{2}(A + A^T)$ are both symmetric. [**6**]

(c) If we take $B = \frac{1}{2}(A + A^T)$, then prove $(A - B)^T = B - A$. [**5**]

(d) Are the matrices $A^T A$ and AA^T always equal? Either prove this result or state a counter-example. [**4**]

(e) Prove that if A is invertible, then so is $A^T A$. [**6**]

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Solutions:

(a) **Similar to examples seen in lecture notes.**

We have

$$C^T = \begin{pmatrix} 2 & -3 & 5 \\ 1 & 1 & -4 \\ -9 & 2 & 0 \end{pmatrix} .$$

We use this to compute the product

$$C^T C = \begin{pmatrix} 2 & -3 & 5 \\ 1 & 1 & -4 \\ -9 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -9 \\ -3 & 1 & 2 \\ 5 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 38 & -21 & -24 \\ -21 & 18 & -7 \\ -24 & -7 & 85 \end{pmatrix}$$

and the linear combination

$$\frac{1}{2}(C + C^T) = \frac{1}{2}\left(\begin{pmatrix} 2 & 1 & -9 \\ -3 & 1 & 2 \\ 5 & -4 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 5 \\ 1 & 1 & -4 \\ -9 & 2 & 0 \end{pmatrix}\right) = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & -1 \\ -2 & -1 & 0 \end{pmatrix} .$$

(b) **Unseen proof.**

By Theorem 7.2.3. d), we have

$$(A^T A)^T = A^T (A^T)^T .$$

Note however, that $(A^T)^T = A$ (Theorem 7.2.3 a)), and therefore

$$(A^T A)^T = A^T A .$$

Hence, the matrix $A^T A$ is equal to its transpose, and is by definition symmetric. For the second matrix, Theorem 7.2.3 b) gives us

$$\left(\frac{1}{2}\left(A + A^T\right)\right)^T = \frac{1}{2}\left(A + A^T\right)^T .$$

We now invoke the result of Theorem 7.2.3 c) to get

$$\left(\frac{1}{2}\left(A + A^T\right)\right)^T = \frac{1}{2}\left(A^T + (A^T)^T\right) = \frac{1}{2}\left(A^T + A\right) ,$$

again using Theorem 7.2.3 a). By the commutativity of matrix addition,

$$\left(\frac{1}{2}\left(A + A^T\right)\right)^T = \frac{1}{2}\left(A + A^T\right) ,$$

giving the result.

(c) **Algebraic manipulation.**

In the previous part, we proved that B is symmetric, and so $B^T = B$. Hence,

$$(A - B)^T = A^T - B^T = A^T - B \, .$$

We can now use the definition of B to obtain

$$\begin{aligned}(A - B)^T &= A^T - \frac{1}{2}\left(A + A^T\right) \\ &= -\frac{1}{2}A + \left(A^T - \frac{1}{2}A^T\right) \\ &= \left(\frac{1}{2}A - A\right) + \frac{1}{2}A^T \\ &= \frac{1}{2}\left(A + A^T\right) - A \\ &= B - A \, .\end{aligned}$$

(d) **Counter-example using definitions given in lecture notes.**

The matrices A^TA and AA^T are, in general, not equal. Indeed, take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \, .$$

We have

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \, ,$$

whereas

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \, .$$

It is clear that in this example, $A^TA \neq AA^T$, and so the conjecture is disproved.

(e) **Unseen proof.**

Firstly, since A is invertible, then by Theorem 7.2.4, so is A^T and

$$(A^T)^{-1} = (A^{-1})^T \, .$$

Secondly, we use Theorem 7.1.19 to show that since A^T and A are both invertible, so is their product A^TA, with

$$(A^TA)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T \, .$$

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Question 4 [25 marks].
Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 7 & -5 \\ 3 & 4 & 9 \end{pmatrix} .$$

(a) Find elementary matrices E_1, E_2, E_3 such that $U = E_3E_2E_1A$, where U is an upper triangular matrix. [**8**]

(b) Evaluate the determinant of A and state whether A is invertible. [**7**]

(c) Evaluate the determinant of the following matrix: [**10**]

$$B = \begin{pmatrix} 7 & 1 & -1 & -4 \\ 8 & 1 & -1 & -4 \\ 5 & 0 & 14 & -10 \\ 9 & 3 & 4 & 9 \end{pmatrix} .$$

Solutions:

(a) **Similar to examples seen in tutorials.**

We can begin the process of Gauss-Jordan Inversion by swapping the second and third rows of A,

$$E_1 A = \begin{pmatrix} 1 & -1 & -4 \\ 3 & 4 & 9 \\ 0 & 7 & -5 \end{pmatrix} ,$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

Next, we subtract 3 times the first row from the second, giving

$$E_2 E_1 A = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 7 & 21 \\ 0 & 7 & -5 \end{pmatrix} ,$$

where

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Finally, we subtract the second row from the third, to get

$$E_3 E_2 E_1 A = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 7 & 21 \\ 0 & 0 & -26 \end{pmatrix} ,$$

where

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} .$$

The resulting matrix $E_3 E_2 E_1 A$ is upper triangular, hence $U = E_3 E_2 E_1 A$.

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(b) **Use of results seen in lectures.**

By Theorem 8.3.12, we have

$$\det(U) = \det(E_3)\det(E_2)\det(E_1)\det(A) \; .$$

Theorem 8.3.11 gives us

$$\begin{aligned} \det(E_1) &= -1 \\ \det(E_2) &= 1 \\ \det(E_3) &= 1 \; , \end{aligned}$$

since E_1 is a Type I elementary matrix, whereas E_1 and E_2 are both Type III. Since U is upper triangular, we can also use Theorem 8.2.8 to compute its determinant as the product of its diagonal entries, giving

$$\det(U) = (1)(7)(-26) = -182 \; .$$

In summary, we have found that

$$-182 = (1)(1)(-1)\det(A) \; ,$$

and thus, $\det(A) = 182$. As this determinant is non-zero, the matrix A is invertible.

(c) **Similar to examples seen in lecture notes.**

We can take a Cofactor expansion down the first column, to get

$$\det(B) = \begin{vmatrix} 7 & 1 & -1 & -4 \\ 8 & 1 & -1 & -4 \\ 5 & 0 & 14 & -10 \\ 9 & 3 & 4 & 9 \end{vmatrix}$$
$$= 7\begin{vmatrix} 1 & -1 & -4 \\ 0 & 14 & -10 \\ 3 & 4 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & -1 & -4 \\ 0 & 14 & -10 \\ 3 & 4 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 & -4 \\ 1 & -1 & -4 \\ 3 & 4 & 9 \end{vmatrix} - 9\begin{vmatrix} 1 & -1 & -4 \\ 1 & -1 & -4 \\ 0 & 14 & -10 \end{vmatrix} .$$

we note that the first two sub-determinants computed above are equal. We also note that since the latter two sub-determinants each contain a repeated row, their value is equal to zero. Hence,

$$\det(B) = -\begin{vmatrix} 1 & -1 & -4 \\ 0 & 14 & -10 \\ 3 & 4 & 9 \end{vmatrix} .$$

Next, we note that the determinant on the right-hand side of the above equality differs from the determinant computed in the previous part only by a factor of 2 in the second row. Hence, by Theorem 8.3.1,

$$\begin{vmatrix} 1 & -1 & -4 \\ 0 & 14 & -10 \\ 3 & 4 & 9 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -4 \\ 0 & 7 & -5 \\ 3 & 4 & 9 \end{vmatrix} = (2)(182) = 364 .$$

Finally, by the equality shown above, we have

$$\det(B) = -\begin{vmatrix} 1 & -1 & -4 \\ 0 & 14 & -10 \\ 3 & 4 & 9 \end{vmatrix} = -364 .$$

End of Paper.

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