Late-Summer Examination period 2017

MTH6127
Metric Spaces and Topology

Duration: 2 hours

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Examiners: M. Farber

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In this examination the symbol $\mathbb{R}$ denotes the sets of real numbers.

Question 1. [4 marks]

(a) Give the definition of *a metric space* (X,d). **[2]**

[Bookwork] A metric space is a pair (X,d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function satisfying:

M1: For $x,y \in X$ one has $d(x,y) \geqslant 0$ and $d(x,y) = 0$ iff $x = y$.

M2: $d(x,y) = d(y,x)$.

M3: $d(x,y) \leqslant d(x,z) + d(z,y)$ for any $x,y,z \in X$.

(b) Explain what it means for a subset $U \subseteq X$ in a metric space to be *open*. **[2]**

[Bookwork] A subset $U \subseteq X$ is open if for any $x \in U$ there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$.

Question 2. [10 marks]

(a) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a metric space X *converges* to a point $x_0 \in X$? **[2]**

[Bookwork] We say that a sequence $x_n \in X$ converges to a point $x_0 \in X$ if for any $\varepsilon > 0$ there exists $N > 0$ such that for any $n > N$ one has $d(x_n, x_0) < \varepsilon$.

(b) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a topological space X *converges* to a point $x_0 \in X$? **[2]**

[Bookwork] We say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a topological space X converges to a point x_0 if for any open set U containing x_0 there is N such that $x_n \in U$ for all $n \geqslant N$.

(c) Let X be a metric space. Is it possible that a sequence of points $\{x_n\}_{n\geqslant 1}$, $x_n \in X$ converges to two distinct points $x_0, x_0' \in X$, $x_0 \neq x_0'$? Justify your answer. **[2]**

[Bookwork] The limit of a sequence in a metric space is unique. Indeed, suppose that $x_n \to x_0$ and $x_n \to x_0'$. Take $\varepsilon = d(x_0, x_0')/2 > 0$. Then for all large n the point x_n must lie in $B(x_0,\varepsilon) \cap B(x_0',\varepsilon) = \emptyset$ - contradiction.

(d) Consider $X = \mathbb{R}$ with the finite-complement topology (i.e. when open subsets are complements of the finite subsets). Consider the sequence $x_n = n \in X$ and find all points $x_0 \in X$ such that the sequence $\{x_n\}_{n\geqslant 1}$ converges to x_0. Justify your answer. **[4]**

[Bookwork] In $\mathbb{R}$ with the finite-complement topology the sequence $\{x_n\}$ with $x_n = n$ converges to any number $x_0 \in \mathbb{R}$.

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Question 3. [10 marks]

(a) Explain what is meant for a metric space (X,d) to be *complete* and give an example of a metric space which is not complete. Justify your answer. **[2]**

[Bookwork] We say that a metric space (X,d) is complete if any Cauchy sequence in X converges. The space $X = \mathbb{R} - \{0\}$ with the induced metric is not complete as the sequence $x_n = 1/n$ is Cauchy but does not have a limit in X.

(b) Let X be a metric space and let $F \subseteq X$ be a closed subset. Show that F is complete with respect to the induced metric. **[2]**

[Bookwork] If x_n is a Cauchy sequence in F then it is converges in X (since X is complete) and the limit $x_0 = \lim x_n$ must belong to F since F is closed. Hence any Cauchy sequence in F converges, i.e. F is complete.

(c) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric? Briefly justify your answer.

(i) $\{3^n; n = 1,2,\dots\}$, **[2]**

[Seen similar] This set is complete as a closed subset of a complete metric space $\mathbb{R}$.

(ii) $\{3^{-n}; n = 1,2,\dots\}$, **[2]**

[Seen similar] This set is not complete since it is a subset which is not closed in $\mathbb{R}$.

(iii) $\{3^{-n}; n = 1,2,\dots\} \cup \{0\}$. **[2]**

[Seen similar] This set is complete as a closed subset of a complete metric space $\mathbb{R}$.

Question 4. [10 marks]

(a) Define the sup metric on the set $C[0,\pi]$ of all real continuous function on the closed interval $[0,\pi]$. **[3]**

[Bookwork] For $f, g \in C[0,\pi]$ we set

$$d(f,g) = \max_{t\in[0,\pi]} |f(t) - g(t)|.$$

We use the fact that the function $f - g$ is continuous on the closed interval and hence it is bounded and attains its maximum and minimum.

(b) Is $C[0,\pi]$ complete? (No proof is required.) **[3]**

[Bookwork] The space $C[0,\pi]$ is complete (it was proven in lectures).

(c) Decide whether the sequence of functions

$$f_n(x) = \sin(nx), \quad x \in [0,\pi],$$

converges in $C[0,\pi]$ with respect to the sup metric. **[4]**

[Seen] Set $x_0 = \pi/2$. Then for any even n one has $f_n(x_0) = \sin(nx_0) = 0$ and for any odd n one has $f_n(x_0) = \sin(nx_0) = \pm 1$ (it is 1 if $n \equiv 1 \mod 4$ and -1 if $n \equiv 3 \mod 4$). Hence we see that the sequence of real numbers $f_n(x_0)$ does not converge for $x_0 = \pi/2$ and hence the sequence of functions $\{f_n\}$ does not converge in $C[0,\pi]$.

Question 5. [23 marks]

(a) Give the $\varepsilon - \delta$ definition of continuity of a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y). **[3]**

[Bookwork] A map $f : X \to Y$ is continuous if for any $x \in X$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

(b) Show that if a map $f : X \to Y$ is continuous then for any open set $U \subseteq Y$ the preimage $f^{-1}(U) \subseteq X$ is open. **[4]**

[Bookwork] Suppose that $f : X \to Y$ is continuous and $x \in f^{-1}(U)$ where $U \subseteq Y$ is open. Then $f(x) \in U$ and there exists $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$. Let $\delta > 0$ be the number given by the definition of continuity, see above. Then

$$f(B(x,\delta)) \subseteq B(f(x),\varepsilon) \subseteq U$$

i.e. $B(x,\delta) \subseteq f^{-1}(U)$. This shows that $f^{-1}(U)$ is open.

(c) Give an example of a non-constant continuous map $f : \mathbb{R} \to \mathbb{R}$ and an open subset $U \subseteq \mathbb{R}$ such that the image $f(U) \subseteq \mathbb{R}$ is not open. **[5]**

[Unseen] Set $f(x) = x^2$ for $x \in \mathbb{R}$. The image of the open interval $(-1,1)$ is $[0,1)$, not open.

(d) Show that if a map $f : X \to Y$ is continuous then for any closed set $F \subseteq Y$ the preimage $f^{-1}(F) \subseteq X$ is closed. **[4]**

[Bookwork] If $F \subseteq Y$ is closed then the complement F^c is open and

$$(f^{-1}(F))^c = f^{-1}(F^c)$$

is open, hence $f^{-1}(F)$ is closed.

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(e) Is it true that the image of a closed set under a continuous map is closed? Explain your answer. **[7]**

[Unseen] Consider the continuous map $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = e^{-x}$. The set $F = \{1, 2, 3, \dots\}$ is closed and its image $f(F) = \{e^{-1}, e^{-2}, e^{-3}, \dots\}$ is not closed. Hence it is not true in general that the image of a closed set under a continuous map is closed.

Question 6. [17 marks]

(a) What is meant by *an open cover* of a topological space? **[2]**

[Bookwork] An open cover of X is a family of open sets $\{U\}$, where $U \subset X$, such that the union $\cup\{U\}$ equals X

(b) When do we say that a topological space is *compact*? **[3]**

[Bookwork] We say that a topological space X is compact if any open cover of X has a finite subcover.

(c) Which of the following subsets of the real line $\mathbb{R}$ are compact? Briefly justify your answer:

(i) $\mathbb{R}$; **[3]**

[Bookwork] Not compact as it is not bounded.

(ii) $[2,3]$; **[3]**

[Bookwork] Compact; it is a closed and bounded subset of the real line.

(iii) $(2,3)$; **[3]**

[Bookwork] Not compact as it is not closed.

(iv) $[2,\infty)$; **[3]**

[Bookwork] Not compact as it is not bounded.

Question 7. [26 marks]

(a) State the contraction mapping theorem. **[5]**

[Bookwork] Let $f : X \to X$ be a contraction of a complete metric space X. Then f has a unique fixed point $x_0 \in X$, i.e. $f(x_0) = x_0$.

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(b) Consider $\mathbb{R}^2$ with the d_1-metric, i.e. $d_1(v,v') = |x-x'|+|y-y'|$ where $v=(x,y)$ and $v'=(x',y')$. Is this metric space complete? Justify your answer. **[5]**

[Bookwork] This metric space is complete. If $v_n = (x_n, y_n)$ is a Cauchy sequence then for any $\varepsilon > 0$ there exists N such that $|x_n - x_m| + |y_n - y_m| < \varepsilon$ for $n,m > N$; hence each of the sequences $\{x_n\}$ and $\{y_n\}$ is Cauchy and hence converges in $\mathbb{R}$. If x_0 and y_0 are their limits then v_n converges to $v_0 = (x_0, y_0)$ in the d_1-metric.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(v) = (\frac{1}{3}y, \frac{1}{2}(x+1)),$$

where $v=(x,y)$. Show that f is a contraction with respect to the d_1-metric. **[10]**

[Unseen] Let $v=(x,y)$ and $v'=(x',y')$. Then $f(v) = (\frac{1}{3}y, \frac{1}{2}(x+1))$ and $f(v') = (\frac{1}{3}y', \frac{1}{2}(x'+1))$. We obtain

$$d_1(f(v), f(v')) = \frac{1}{3}|y-y'| + \frac{1}{2}|x-x'| \leqslant \frac{1}{2}d_1(v,v').$$

Hence we see that f is a contraction with coefficient $\alpha = 1/2$.

(d) Find the fixed point of f. **[6]**

[Unseen, seen similar] To find the fixed point we solve the system of equations

$$\begin{aligned} \tfrac{1}{3}y &= x, \\ \tfrac{1}{2}(x+1) &= y. \end{aligned}$$

We find $(x,y) = (1/5, 3/5)$.

End of Paper.

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