Solutions for MATH5125 May, 2023

1. A life assurance company sells a life annuity which pays out £1,000 per year to one individual aged 60 for £11,280 (one payment at the signing of the contract). The annuity is paid annually in arrears. The valuation basis for this question is as follows: Mortality follows AM92 ultimate, interest: 6% per annum.

a) Express the profit random variable at the issuance of this policy.

b) Calculate the expected present value of the life insurer's profit.

c) Calculate the standard deviation of the present value of the life insurer's profit.

d) Calculate the probability that the present value of the life insurer's profit on this contract will be positive.

e) Do you have any concerns about issuing this policy, from a profit viewpoint? Briefly explain your answer.

d) Suppose the life annuity was paid annually in advance. Without doing any more calculations, would the standard deviation of the present value of the profit be less than, equal to or greater than the standard deviation calculated in (b)? Explain your answer.

Solutions

a)

$$\begin{aligned} P &= \text{Income - Benefits outgo} \\ &= 11280 - 1000 a_{\overline{K_{60}}|} \end{aligned}$$

b)

From the table $\ddot{a}_{60} = 11.891$ hence $a_{60} = 10.891$

$$\begin{aligned} EPV\left(P\right) &= 11280 - 1000 a_{60} \\ &= 11280 - 1000 \times 10.891 = 389 \end{aligned}$$

c)

$$\begin{aligned}
Var(P) &= 1000 \times Var\left(a_{\overline{K_{60}|}}\right) \\
&= 1000 \times Var\left(\frac{1 - v^{K_{60}}}{i}\right) \\
&= 1000 \times \frac{1}{(iv)^2} Var\left(v^{K_{60}+1}\right) \\
&= 1000 \times \frac{1}{d^2}\left({}^2A_{60} - A_{60}^2\right) \\
&= 1000 \times \frac{0.14098 - (0.32692)^2}{(1 - 1.06^{-1})^2} \\
&= 10644
\end{aligned}$$

$$StDev(P) = \sqrt{Var(P)} = 3263$$

d)

$$\begin{aligned}
\Pr(P > 0) &= \Pr\left(1000\left(11.28 - a_{\overline{K_{60}|}}\right) > 0\right) \\
&= \Pr\left(11.28 - a_{\overline{K_{60}|}} > 0\right) \\
&= \Pr\left(v^{\overline{K_{60}|}} > 1 - 11.28i\right) \\
&= \Pr\left(K_{60} < -\frac{\ln(1 - 11.28i)}{\ln(1+i)}\right) \\
&= \Pr(K_{60} < 19.384) \\
&= \Pr(K_{60} \le 19) \\
&= \Pr(K_{60} + 1 \le 20) \\
&= {}_{20}q_{60} \\
&= 1 - {}_{20}p_{60} = 1 - \frac{l_{80}}{l_{60}} \\
&= 1 - \frac{5266.4604}{9287.2164} = 0.4329
\end{aligned}$$

d) As calculated in part (c), there is a 56.71% chance of a loss.

The maximum possible loss is £1000 annuity in arrears for life: PV is $= 1000\left(\frac{1}{1-\frac{1}{1+i}} - 1\right) = \frac{1000}{i} =$ £16666.67 which gives a loss of£5386.67. The maximum possible profit is £11280, corresponding to the person dying before age 61. Therefore, the present value of the profit lies in the interval $(-5386.67, 11280]$.

The expected present value of the profit is positive at £389. However, the standard deviation is almost ten times as large, at £3263. This suggests that the distribution of the present value of the profit is spread out.

e) It would not change at all. The only difference would be a payment at age 60 with probability 1, which would not affect the variance (or standard deviation).

2. A life assurance company issues a 20-year term, insurance policy to a life aged 40, with sum insured £100,000 payable immediately on death. Level premiums are payable monthly in advance throughout the term. The commissions are 10% of each premium payment (incurred at the premium payment times). Mortality follows AM92 ultimate life table with Uniform distribution of deaths (UDD) assumption between integer years. $i = 6\%$.

a) Write down an expression for the gross loss at issue random variable.

c) Write down the equation of value for this policy.

b) Calculate the gross monthly premium.

Solution:

a)

$$L_0^g = PV \text{ of benefits } + \ PV \text{ of expenses} - PV \text{ of premiums}$$

Let P be the monthly premium.

PV of benefits:

$$100,000\left[\exp\left(-\delta T_{40}\right) \times 1\left(T_{40} \leq 20\right)\right]$$

- T_{40} be the future life time random variable of a life aged 40 of deaths
- $\left[\exp\left(-\delta T_{40}\right) \times 1\left(T_{40} \leq n\right)\right]$ is the PV of a continuous term insurance with unit benefit.
- $1\left(T_{40} \leq 20\right)$ is an indicator function which accounts for whether the insured dies within 20 years

PV of expenses:

$$0.1P\ddot{a}^{(12)}_{\overline{min(K_{40}+1),20|}}$$

PV of premiums:

$$P\ddot{a}^{(12)}_{\overline{min(K_{40}+1),20|}}$$

Hence

$$L_0^g = 100,000\left[\exp\left(-\delta T_{40}\right) \times 1\left(T_{40} \leq 20\right)\right] - 0.9P\ddot{a}^{(12)}_{\overline{min(K_{40}+1),20|}}$$

b) The equation of value:

$$E\left(L_0^g\right) = 0$$

$$100,000\bar{A}^1_{40:\overline{20}|} = 0.9P\ddot{a}^{(12)}_{40:\overline{20}|}$$

c)

$$P = \frac{100,000\bar{A}^1_{40:\overline{20}|}}{0.9\ddot{a}^{(12)}_{40:\overline{20}|}} = 265.3716$$

We used the following calculations:

$$\bar{A}^1_{40:\overline{20}|} = \frac{i}{\delta}A^1_{40:\overline{20}|}$$

$$A^1_{40:\overline{20}|} = A_{40:\overline{20}|} -_{20} E_{40}$$

$$_{20}E_{40} = v^{20}\ {}_{20}p_{40} = 0.2938$$

since ${}_{20}p_{40} = \frac{l_{60}}{l_{40}} = \frac{9287.2164}{9856.2863} = 0.942263$ and $v^{20} = \left(\frac{1}{1.06}\right)^{20}$

$$A_{40:\overline{20}|} = 0.32088$$

from the AM92 table

Hence

$$A^1_{40:\overline{20}|} = 0.32088 - 0.2938 = 0.027078$$

$$\begin{aligned} \bar{A}^1_{40:\overline{20}|} &= \frac{0.06}{\ln(1.06)}A^1_{40:\overline{20}|} \\ &= 0.027882 \end{aligned}$$

$$\begin{aligned} \ddot{a}^{(12)}_{40:\overline{20}|} &\simeq \ddot{a}_{40:\overline{20}|} - \frac{12-1}{24}(1 - v^{20}\ {}_{20}p_{40}) \\ &= 11.998 - \frac{11}{24}(1 - 0.2938) \\ &= 11.67433 \end{aligned}$$

3. On 1 January 2017, an insurer issued whole life assurances to lives then aged exactly 65. The number of policies in force on 1 January 2022 was 1,900, the number in force on 1 January 2023 was 1,867. The sum insured was £100,000 payable at the end of the year of death. The level premiums are payable annually in advance for the whole of life. Assume that death is the only cause of policy termination, and that the insurer holds net premium reserves for these contracts. Mortality follows AM92 ultimate life table and interest is 4% per annum.

a) Calculate the net premium for these policies

b) Calculate the death strain at risk for the policy in the year commencing on 1 January 2022

c) Calculate the mortality profit for the policy in the year commencing on 1 January 2022.

Solution

a) From the equivalence principle:

$$\begin{aligned} P &= \frac{100,000\ A_{65}}{\ddot{a}_{65}} = \\ & \frac{100,000 A_{65}}{\ddot{a}_{65}} \\ &= 100,000 \times \frac{0.52786}{12.276} \\ & 4,299.935 \end{aligned}$$

b) For calculating the profit at the end of 2022 we need to calculate the reserve at the end on 2022

$$\begin{aligned} {}_6V &= 100,000 A_{71} - P\ddot{a}_{71} \\ &= 100,000 \times 0.61548 - 4,299.935 \times 9.998 \\ &= 18,557.25 \end{aligned}$$

$$DSAR = 100,000 - 18,557.25 = 81,442.75$$

c)

$$\begin{aligned} EDS &= 1,900 \times q_{70} \times DSAR \\ 1,900 \times 0.024783 \times 81,442.75 &= 3,834,972 \end{aligned}$$

During the policy year, 33 people died, so the actual death strain was:

$$ADS = 33 \times DSAR = 2,687,611$$

This gives a mortality profit of $EDS - ADS = 1,147,341$

4. Shaun and Riley are independent lives, both aged 60. They purchase an insurance policy which provides £200,000 payable at the end of the year of Shaun's death, provided that Shaun dies after Riley. Annual premiums are payable in advance throughout Shaun's lifetime. You are given $\ddot{a}_{60} = 15.632$, $\ddot{a}_{70} = 11.762$ and $\ddot{a}_{60:60} = 14.090$, $\ddot{a}_{70:70} = 9.766$ and $i = 4\%$ per annum.

a) Calculate the net annual premium of this policy.

b) Calculate the net premium policy value after 10 years if only Shaun is alive.

b) Calculate the net premium policy value after 10 years if both Shaun and Riley are alive

d) Explain why the insurer cares whether both are alive or only Shaun is alive.

Solution

a) Let P represent the annual premium.

From the Tables: $\ddot{a}_{60} = 15.632$ $\ddot{a}_{60:60} = 14.090$. Then

$A_{60} = (1 - d\ddot{a}_{60}) = \left(1 - \frac{0.04}{1.04} \times 15.632\right) = 0.398769$

$A_{60:60} = (1 - d\ddot{a}_{60:60}) = \left(1 - \frac{0.04}{1.04} \times 14.090\right) = 0.458076$

The EPV of the premiums is:

$$P\ddot{a}_{60} = 15.632P$$

The EPV of the death

benefit is (initially I will use letters S,R for the lives to avoid confusion):

$$200{,}000 A_{\overset{2}{S}:R}$$

Now, $A_S = A_{\overset{1}{S}:R} + A_{\overset{2}{S}:R}$ so $A_{\overset{2}{S}:R} = A_S - A_{\overset{1}{S}:R}$.

Replacing S and R by the age 60 and noting by symmetry that $A_{\overset{1}{60}:60} = \frac{1}{2}A_{60:60}$ we can say that $A_{\overset{2}{60}:60} = A_{60} - \frac{1}{2}A_{60:60} = 0.16973$.

The EPV of the death benefit is:

$$200{,}000 \times A_{\overset{2}{60}:60} = 33{,}946.1539$$

Hence by equivalence principle

$$P = \frac{33{,}946.15385}{15.632} = 2171.7810$$

b) $\ddot{a}_{70} = 11.762$ and $\ddot{a}_{70:70} = 9.766$, hence $A_{70} = 0.5476$ and $A_{70:70} = 0.2354$

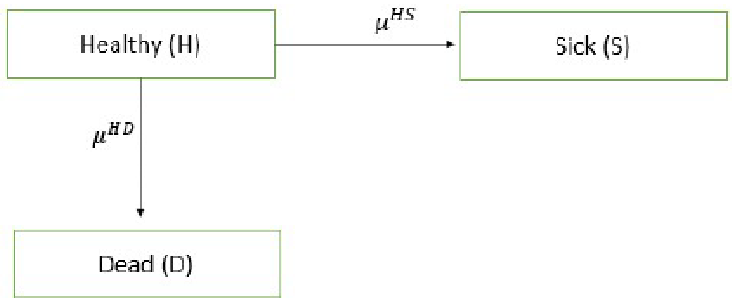
If Shaun is alive but Riley not the policy value is:

$${}_{10}V = 200{,}000A_{70} - P\ddot{a}_{70} = 83{,}980.94146$$

c) If both are alive:

$${}_{10}V = 200{,}000\left(A_{70} - \frac{1}{2}A_{70:70}\right) - P\ddot{a}_{70} = 60{,}438.6337$$

d) The first policy value (at b) is higher than that at c) as the likelihood of paying the death benefit is higher in this case as Riley is already dead. In the second case there is still a positive probability that Riley dies after Shaun which means that no death benefit is paid. Hence, the insurer needs to build more reserves if only Shaun is alive.



5. A population of healthy people over the year of age 50 to 51 is subject to a constant force of decrement due to sickness of 0.08 per annum, and a constant force of mortality of 0.002 per annum. Assume that a double decrement model is used.

a) Calculate the probability that a healthy person aged exactly 50 will still be healthy at exact age 51.

b) Calculate the probability that a healthy person aged exactly 50 will die due to any cause other than from sickness before exact age 51

c) Calculate the independent probability of a life aged exactly 50 dying before exact age 51.

d) Explain why the probability found at c) is unlikely to be a realistic estimate of the probability of a healthy life aged exactly 50 dying before exact age 51

e) Propose a more suitable model to estimate the probability required in part d).

Solution

a) The probability of staying healthy is:

$$p_{50}^{HH} = \exp\left(-(0.08 + 0.002\right) = 0.921272$$

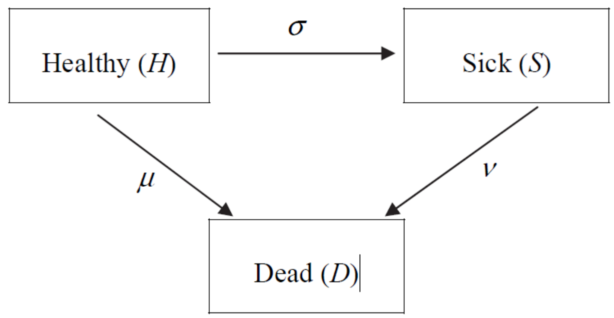
b) The dependent probability of leaving the population through death is:

$$p_{50}^{HD} = \frac{0.002}{0.082}\left(1 - \exp\left(-0.082\right)\right) = 0.001920$$

c) The independent probability of dying is:

$$q_x^{*D} = 1 - \exp\left(-0.002\right) = 0.001998$$

d) The calculation in part c) is unlikely to equal the probability of a healthy life dying over the year, because it makes the implicit assumption that the force of mortality is the same for a sick life as for a healthy life. A person who becomes sick during the year is likely to have a considerably higher force of mortality for the rest of that year, after becoming sick, and so the actual probability



of a healthy life dying by the end of the year should be higher. The double decrement model that we have used is not appropriate in this case

e) A more suitable models should have been a multiple state model where mortality is assumed to be different for healthy and sick lives

IFoA mapping

Question 1: Single decrement models and Pricing and reserving

- CM1 Syllabus Objective: 4.2, 6.1
- Standard question (similar to class exercises) for the first three parts, and more difficult questions for the last two parts, which test understanding.

Question 2: Gross premium calculations, Pricing and reserving

- CM1 Syllabus Objective: 6.1, 6.2
- Medium level of difficulty (students need to be able to move from continous, annual and monthly payments structure).

Question 3: Mortality profit

- CM1 Syllabus Objective: 6.3
- Standard question (similar to class exercises)

Question 4: Joint life and Pricing and reserving

- Standard question (similar to class exercises) - last part more difficult as it tests further understanding.
- CM1 Syllabus Objective: 5.1, 5.2

Question 5: Multiple decrement and multiple state models

- CM1 Syllabus Objective: 5.2, 5.3
- Standard question (similar to class exercises) for the first three parts, and medium difficulty level for the last two parts, which test understanding.

All guestions are unseen.