Vectors & Matrices

Solutions to Problem Sheet 10

1. (i) We can perform a Cofactor Expansion down the first column to obtain

$$\begin{vmatrix} 3 & 1 & -2 \\ -4 & 5 & 4 \\ 1 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 5 & 4 \\ 2 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix} .$$

We can now use the definition of a 2×2 determinant to evaluate

$$\begin{vmatrix} 5 & 4 \\ 2 & -1 \end{vmatrix} = (5)(-1) - (4)(2) = -13 ,$$

$$\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = (1)(-1) - (-2)(2) = 3 ,$$

$$\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix} = (1)(4) - (-2)(5) = 14 ,$$

and find

$$\begin{vmatrix} 3 & 1 & -2 \\ -4 & 5 & 4 \\ 1 & 2 & -1 \end{vmatrix} = 3(-13) - (-4)(3) + (1)(14) = -13 .$$

(ii) We note that the first row of the determinant

$$\begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix}$$

differs from the first row of the determinant in part (i) only by a factor of 2. Hence, we use Theorem 8.3.1 to obtain

$$\begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} = (2)\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} .$$

This new determinant differs from the determinant in part (i) by only a swap of the second and third rows. We therefore invoke the result of Theorem 8.3.1 a) to find

$$\begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} = (2) \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} = (2)(-1) \begin{vmatrix} 3 & 1 & -2 \\ -4 & 5 & 4 \\ 1 & 2 & -1 \end{vmatrix} .$$

This final determinant is equal to the one evaluated in part (i), the value of which we know to be -13, hence

$$\begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} = (2)(-1)(-13) = 26 .$$

(iii) We again proceed by a Cofactor Expansion. We will choose to expand down the first column for two reasons:

- It is clear that the sub-determinants formed by eliminating the first column and first row, and the first column and second row, are both equal to the determinant given in part (ii), the value of which we already know.
- All other sub-determinants formed by eliminating the first column and a single row have a repeated row, which we can show always results in a value of zero.

We have

$$\begin{vmatrix} 3 & 6 & 2 & -4 \\ 2 & 6 & 2 & -4 \\ -7 & 1 & 2 & -1 \\ -1 & -4 & 5 & 4 \end{vmatrix} = (3) \begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} - (2) \begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} + (-7) \begin{vmatrix} 6 & 2 & -4 \\ 6 & 2 & -4 \\ -4 & 5 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 6 & 2 & -4 \\ 6 & 2 & -4 \\ -1 & 2 & -1 \end{vmatrix} .$$

As discussed above, we already have

$$\begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} = 26$$

from part (ii).

Next, consider a determinant with at least two equal rows, e.g.

$$\begin{vmatrix} 6 & 2 & -4 \\ 6 & 2 & -4 \\ -4 & 5 & 4 \end{vmatrix} ,$$

By Theorem 8.3.1 c), we can subtract the second row from the first, and the resulting determinant will be equal to the original, i.e.

$$\begin{vmatrix} 6 & 2 & -4 \\ 6 & 2 & -4 \\ -4 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 6 & 2 & -4 \\ -4 & 5 & 4 \end{vmatrix} .$$

A Cofactor Expansion across the first row of this right-hand determinant would always give terms containing factors of zero, and so regardless of the value of each respective cofactor, the resulting determinant will be equal to zero. We have now found values for all four subdeterminants generated from our determinant, and can compute

$$\begin{vmatrix} 3 & 6 & 2 & -4 \\ 2 & 6 & 2 & -4 \\ -7 & 1 & 2 & -1 \\ -1 & -4 & 5 & 4 \end{vmatrix} = (3)(26) - (2)(26) + (-7)(0) - (-1)(0) = 26 .$$

2. Let E_i be the Type II elementary matrix, the effect of which is to multiply the i^{th} row of a matrix by a factor of α, i.e.

$$E_1 = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} , \ E_2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} , \dots , \ E_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix} .$$

By Theorem 8.3.11, we see that $\det(E_i) = \alpha$, for each $i \in \{1, \dots, n\}$. By definition, αA is equal to the matrix A with each one of its entries multiplied by a factor of α. Since the effect of left-multiplying A by E_i is to rescale the i^{th} row of A by α, we have

$$\alpha A = E_1 E_2 \dots E_n A .$$

We can therefore use the multiplicativity of determinants (Theorem 8.3.12) to show

$$\begin{aligned}\det(\alpha A) &= \det(E_1 E_2 \dots E_n A)\\ &= \det(E_1)\det(E_2 \dots E_n A)\\ &= \alpha \det(E_2 \dots E_n A)\ ,\end{aligned}$$

since $\det(E_1) = \alpha$. We can replicate this idea with all other elementary factors in the right-hand determinant to obtain

$$\begin{aligned}\det(\alpha A) &= \alpha \det(E_2 \dots E_n A)\\ &= \alpha \det(E_2)\det(E_3 \dots E_n A)\\ &= \alpha^2 \det(E_3 \dots E_n A)\\ &= \dots\\ &= \alpha^n \det(A)\ ,\end{aligned}$$

giving the result.

3. By Theorem 8.3.9 b), if we multiply the second column of the determinant by a factor of a, and multiply the entire determinant by a factor of $\frac{1}{a}$, then the overall value will not change. Therefore,

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = \frac{1}{a}\begin{vmatrix} x_1 & ax_1^2 \\ x_2 & ax_2^2 \end{vmatrix}\ .$$

Since x_1 and x_2 both solve the equation $ax^2 + bx + c = 0$, we have

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = \frac{1}{a}\begin{vmatrix} x_1 & -bx_1 - c \\ x_2 & -bx_2 - c \end{vmatrix}\ .$$

Next, as a result of Theorem 8.3.9 c), it is clear that adding b times the first column to the second will not change the value of the determinant, giving

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = \frac{1}{a}\begin{vmatrix} x_1 & -c \\ x_2 & -c \end{vmatrix}\ .$$

Finally, Theorem 8.3.9 b) tells us that we can extract the factor of $-c$ present in all terms in the second column, and bring it outside the determinant as

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = -\frac{c}{a} \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} .$$

We now use the formula of a 2×2 determinant to obtain

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = -\frac{c((x_1)(1) - (1)(x_2))}{a} = \frac{c(x_2 - x_1)}{a} .$$

All that remains is to evaluate $x_2 - x_1$ in terms of the coefficients a, b, c. Since $x_1 < x_2$, then by the quadratic formula,

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} ,$$

and so

$$\begin{aligned} x_2 - x_1 &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \\ &= \frac{-b + \sqrt{b^2 - 4ac} - (-b - \sqrt{b^2 - 4ac})}{2a} \\ &= \frac{2\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{\sqrt{b^2 - 4ac}}{a} . \end{aligned}$$

Combining this with our formulation for the determinant above, we have

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = \frac{c\sqrt{b^2 - 4ac}}{a^2} .$$

4. (i) Let $P = (p_{ij})$ and $Q = (q_{ij})$ be permutation matrices of size $n \times n$. The element of the product PQ at index (i, j) is given by

$$\sum_{k=1}^{n} p_{ik} q_{kj} .$$

However, since P is a permutation matrix, as we move along the i^{th} row there is only a single value $k \in \{1, \ldots n\}$ such that $p_{ik} = 1$. For all other values of k, p_{ik} is equal to zero. The same holds for the values q_{kj} down the j^{th} column of Q (although this need not be the same value k that gave the non-zero entry in the i^{th} row of P).

For the i^{th} row of P, let k_1 be the unique index such that $p_{ik_1} = 1$. For the j^{th} column of Q, let k_2 be the unique index such that $q_{k_2 j} = 1$. If $k_1 = k_2$, we have

$$\sum_{k=1}^{n} p_{ik} q_{kj} = p_{ik_1} q_{k_2 j} = (1)(1) = 1 \ ,$$

and hence, the entry of PQ at (i, j) is equal to one. Otherwise, if $k_1 \neq k_2$,

$$\sum_{k=1}^{n} p_{ik} q_{kj} = p_{ik_1} p_{k_1 j} + p_{ik_2} q_{k_2 j} = (1)(0) + (0)(1) = 0 \ ,$$

and the entry of PQ at (i, j) is equal to zero. We now fix a row i of the product PQ, and move along each of the columns. We have already seen that for the row i, there is a unique index k_1 such that $p_{ik_1} = 1$. As we move through each column j, each new value of j gives a corresponding row index k_2 such that $q_{k_2 j} = 1$. Moreover, since permutation matrices have only a single non-zero entry in each of their rows, every possible value of k_2 is attained as the matching row index of j once and only once.

Therefore, as we move across the i^{th} column of PQ, each new value of j will give a new corresponding row index k_2 such that $q_{k_2 j} = 1$, and exactly one of these row indices will have the property that $k_2 = k_1$. Thus, there will be exactly one entry along the i^{th} row of PQ that has a value of 1, and all other entries will have a value of 0. A similar argument holds for each column of PQ. By definition, PQ is therefore a permutation matrix.

(ii) Let $P = (p_{ij})$ be a permutation matrix of size $n \times n$. Its transpose $P^T = (\tilde{p}_{ij})$ is also a matrix of size $n \times n$, with entries $\tilde{p}_{ij} = p_{ji}$. The (i, j) entry of the product PP^T is given by

$$\sum_{k=1}^{n} p_{ik}\tilde{p}_{kj} = \sum_{k=1}^{n} p_{ik}p_{jk} \ .$$

As discussed above, for each row i, there is a unique $k_1 \in \{1, \ldots, n\}$ such that $p_{ik_1} = 1$. For every other value of k, $p_{ik} = 0$. As there exists only a single non-zero value in each column of P, if $i \neq j$, then for any k_2 such that $p_{jk_2} = 1$, we have $p_{ik_2} = 0$. Hence, if $i = j$,

$$\sum_{k=1}^{n} p_{ik}\tilde{p}_{kj} = \sum_{k=1}^{n} p_{ik}p_{jk} = p_{ik_1}p_{jk_1} = (1)(1) = 1 \ .$$

Otherwise, if $i \neq j$,

$$\sum_{k=1}^{n} p_{ik}\tilde{p}_{kj} = \sum_{k=1}^{n} p_{ik}p_{jk} = p_{ik_1}p_{jk_1} + p_{ik_2}p_{jk_2} = (1)(0) + (0)(1) = 0 \ .$$

Therefore, the (i, j) entry of PP^T is equal to 1 if $i = j$, and 0 if $i \neq j$, and so $PP^T = I_n$. By Corollary 7.6.10, the inverse of P exists and $P^{-1} = P^T$.

(iii) By part (ii), for any permutation matrix P, we have $PP^T = I$. We know that for any identity matrix I, $\det(I) = 1$. Moreover, by the multiplicativity property of the determinant (Theorem 8.3.12), we have

$$\det(P)\det(P^T) = \det(PP^T) = \det(I) = 1 \ .$$

Finally, we use Theorem 8.3.8 to show that

$$\det(P^T) = \det(P) \ ,$$

and thus

$$(\det(P))^2 = \det(P)\det(P^T) = 1 \ ,$$

showing that $\det(P)$ must either equal 1 or -1.

(iv) Changing all of the non-zero entries of A to 1, we have

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

The effect of left-multiplying a matrix P by a diagonal matrix $D = (d_{ij})$ is to re-scale the i^{th} row of P by a factor of d_{ii} (the i^{th} entry along the diagonal of D). Thus, if we take

$$D = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 92 \end{pmatrix} ,$$

we then have

$$DP = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 92 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 25 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 19 \\ 92 & 0 & 0 & 0 \end{pmatrix} = A .$$

(v) By part (iii), we know that $\det(P) = \pm 1$. (In fact, direct computation shows that $\det(P) = 1$). Theorem 8.2.8 tells us that the determinant of D is equal to the product of its diagonal entries, hence $\det(D) = 174800$. Since neither of these values are zero, their product is also non-zero, and so, by Theorem 8.3.12,

$$\det(A) = \det(DP) = \det(D)\det(P) \neq 0 .$$

Hence, by Theorem 8.3.5, the matrix A is invertible. As $A = DP$, we can use the result of Theorem 7.1.19 to find

$$A^{-1} = P^{-1}D^{-1} .$$

We know from part (ii) that $P^{-1} = P^T$, and so this is simple to evaluate;

$$P^{-1} = P^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

As the effect of the diagonal matrix D is to simply re-scale the i^{th} row of a matrix by its i^{th} diagonal entry d_{ii}, its inverse is the diagonal matrix with diagonal entries $\frac{1}{d_{ii}}$, i.e.

$$D^{-1} = \begin{pmatrix} \frac{1}{25} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{19} & 0 \\ 0 & 0 & 0 & \frac{1}{92} \end{pmatrix}.$$

Hence,

$$\begin{aligned} A^{-1} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{25} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{19} & 0 \\ 0 & 0 & 0 & \frac{1}{92} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{92} \\ 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{25} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{19} & 0 \end{pmatrix}. \end{aligned}$$