Vectors & Matrices

Problem Sheet 10

1. Evaluate the following determinants:

$$\text{(i)} \quad \begin{vmatrix} 3 & 1 & -2 \\ -4 & 5 & 4 \\ 1 & 2 & -1 \end{vmatrix} \qquad \text{(ii)} \quad \begin{vmatrix} 6 & 2 & -4 \\ 1 & 2 & -1 \\ -4 & 5 & 4 \end{vmatrix} \qquad \text{(iii)} \quad \begin{vmatrix} 3 & 6 & 2 & -4 \\ 2 & 6 & 2 & -4 \\ -7 & 1 & 2 & -1 \\ -1 & -4 & 5 & 4 \end{vmatrix}$$

2. Prove that for any $n \times n$ matrix A and scalar $\alpha \in \mathbb{R}$,

$$\det(\alpha A) = \alpha^n \det(A) \, .$$

3. Let $x_1 < x_2$ be real-valued solutions to a quadratic equation $ax^2 + bx + c = 0$ (with $a, b, c \in \mathbb{R}$ and $a \neq 0$). Show that

$$\begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix} = \frac{c\sqrt{b^2 - 4ac}}{a^2} \, .$$

4. Permutation Matrices are square matrices that have exactly one non-zero entry in each row and each column, and these non-zero entries are all equal to 1.

   (i) Prove that the product of two permutation matrices is a permutation matrix.

   (ii) Prove that the transpose of a permutation matrix is also its inverse.

   (iii) Show that the determinant of a permutation matrix is always equal to 1 or -1.

   (iv) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 25 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 19 \\ 92 & 0 & 0 & 0 \end{pmatrix} .$$

   Let P be the permutation matrix formed by changing all of the non-zero values in A to 1. Find a diagonal matrix D such that $A = DP$.

   (v) Show that A is invertible and find A^{-1}.