MTH5114 **Linear Programming and Games, Spring 2024**

Week 9 Seminar Questions **Viresh Patel**

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1. Use the principle of complementary slackness to determine whether or not $\mathbf{x}^\mathsf{T} = (0, 4, 0, 2)$ is an optimal solution to the following linear program.

$$\begin{aligned} \text{maximize} \quad & 9x_1 + 3x_2 + 5x_3 + 22x_4 \\ \text{subject to} \quad & 2x_1 - x_2 + 2x_3 + 6x_4 \leq 8, \\ & 5x_1 + 3x_2 + x_3 + 2x_4 \leq 16, \\ & 4x_1 + x_2 - x_3 + 3x_4 \leq 12, \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution: To see if $\mathbf{x}$ is optimal, let's try to find a feasible dual solution $\mathbf{y}$ that, together with $\mathbf{x}$, satisfies the complementary slackness conditions.

We see that any such $\mathbf{y}$ must make the 2nd and 4th dual constraints tight, since x_2 and x_4 are non-zero. Additionally, since the 3rd primal constraint is not tight for $\mathbf{x}$, we must have $y_3 = 0$. We get the following system of equations:

$$\begin{aligned} -y_1 + 3y_2 + y_3 &= 3 \\ 6y_1 + 2y_2 + 3y_3 &= 22 \\ y_3 &= 0 \end{aligned}$$

Solving this system of equations, we get a unique solution $\mathbf{y}^\mathsf{T} = (3, 2, 0)$. Clearly $\mathbf{y} \geq \mathbf{0}$ and also satisfies the 2nd and 4th dual constraints. We need to make sure the other two dual constraints hold. They are:

$$\begin{aligned} 2y_1 + 5y_2 + 4y_3 &\geq 9 \\ 2y_1 + y_2 - y_3 &\geq 5 \end{aligned}$$

Both are true, and so we have found a feasible solution $\mathbf{y}$ to the dual that, together with $\mathbf{x}$ satisfies the complementary slackness conditions. It follows that $\mathbf{x}$ must be optimal.

2. Use the principle of complementary slackness to determine whether or not $\mathbf{x}^\mathsf{T} =$

$(0, \frac{5}{6}, \frac{3}{2})$ is an optimal solution to the following linear program.

$$\begin{aligned} \text{maximize} \quad & \tfrac{1}{2}x_1 + x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + 7x_2 + x_3 \leq 8, \\ & x_1 + 3x_2 + 3x_3 \leq 7, \\ & 2x_1 \qquad + 4x_3 \leq 6, \\ & x_1 + 3x_2 + \ x_3 \leq 9, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Converting to standard equation form, we find the following slack for each constraint: $s_1 = \frac{2}{3}$, $s_2 = 0$, $s_3 = 0$, $s_4 = 5$. Since these are all non-negative, $\mathbf{x}$ is feasible. [Note that converting to standard equation form and finding the values of the s_i is just another way of checking feasibility and determining which constraints are tight.] We see that $x_2, x_3 > 0$. Thus, any $\mathbf{y}$ that satisfies complementary slackness, together with $\mathbf{x}$ must make the corresponding dual constraints tight:

$$\begin{aligned} 7y_1 + 3y_2 + 3y_4 &= 1 \\ y_1 + 3y_2 + 4y_3 + y_4 &= 1. \end{aligned}$$

Additionally, $s_1, s_4 > 0$, so the first and fourth primal constraints are not tight and thus any $\mathbf{y}$ that satisfies complementary slackness together with $\mathbf{x}$ must have:

$$\begin{aligned} y_1 &= 0 \\ y_4 &= 0 \end{aligned}$$

Solving these equations, we obtain a unique solution $\mathbf{y}^\mathsf{T} = (0, \frac{1}{3}, 0, 0)$. Clearly $\mathbf{y} \geq \mathbf{0}$, and $\mathbf{y}$ satisfies the second and third dual constraints by construction. For the remaining dual constraint, we have:

$$2 \cdot 0 + \frac{1}{3} \cdot 1 + 2 \cdot 0 + 0 = \frac{1}{3} < \frac{1}{2}.$$

Thus, the only $\mathbf{y}$ that together with $\mathbf{x}$ satisfies complementary slackness is *infeasible* for the dual. It follows that $\mathbf{x}$ is *not* optimal.

Practice Exam Question: A power company operates three power generation plants. One is a wind plant, and the other two consume a combination of Fuel 1 and Fuel 2, emitting carbon dioxide in the process. In addition, all three plants require maintenance. The amount of fuel consumed (in Mg), maintenance required (in person-hours), carbon dioxide (CO_2) emitted (in Mg), and power generated (in MWh) per day of operation is as follows:

Plant	Maintenance Required	Fuel 1 Required	Fuel 2 Required	CO_2 Emitted	Power Produced
1	20	0	0	0	20
2	13	10	15	12	32
3	18	30	40	29	40

Each MWh of power can be sold at £121 and there is no limit on the amount that can be sold. Over its next planning period, the company has 230 person-hours for maintenance, 75 Mg of Fuel 1, and 90 Mg of Fuel 2 available.

(a) Due to environmental regulations, they cannot emit more than 200Mg of CO_2 in this period. The company wants to know how to operate its plants to generate as much revenue as possible. (You may assume that there is no limit on the number of days a plant can operate in this period). Give a linear program that models this problem. You do not need to solve this program.

Solution:

$$\begin{array}{ll} \text{maximize} & 121o_2 \\ \text{subject to} & i_1 = 20x_1 + 13x_2 + 18x_3, \\ & i_2 = 10x_2 + 30x_3, \\ & i_3 = 15x_2 + 40x_3, \\ & o_1 = 12x_2 + 29x_3, \\ & o_2 = 20x_1 + 32x_2 + 40x_3, \\ & i_1 \leq 230, \\ & i_2 \leq 75, \\ & i_3 \leq 90, \\ & o_1 \leq 200, \\ & x_1, x_2, x_3 \geq 0, \\ & i_1, i_2, i_3, o_1, o_2 \text{ unrestricted} \end{array}$$

(b) Suppose now that the company can emit more than 200Mg of CO_2, but now loses £55 of revenue for each Mg emitted after the first 200Mg because it must purchase "CO_2 credits". The other resource constraints remain as stated. The company now wants to know how to operate its plants to generate as much revenue as possible (where revenue should now take account of this extra cost). (Again, you may assume that there is no limit on the number of days that a plant can operate in this period.) Give a linear program that models this problem. You do not need to solve this program.

Solution:

$$\begin{array}{ll} \text{maximize} & 121o_2 - 55o_{1,2} \\ \text{subject to} & i_1 = 20x_1 + 13x_2 + 18x_3, \\ & i_2 = 10x_2 + 30x_3, \\ & i_3 = 15x_2 + 40x_3, \\ & o_1 = 12x_2 + 29x_3, \\ & o_2 = 20x_1 + 32x_2 + 40x_3, \\ & o_1 = o_{1,1} + o_{1,2}, \\ & o_{1,1} \leq 200, \\ & i_1 \leq 230, \\ & i_2 \leq 75, \\ & i_3 \leq 90, \\ & x_1, x_2, x_3, o_{1,1}, o_{1,2} \geq 0, \\ & i_1, i_2, i_3, o_1, o_2 \text{ unrestricted} \end{array}$$

The solution above is obtained by directly modelling the situation. You could come up with an alternative solution by noting that the extra cost z of emitting carbon dioxide is given by $z = \max(0, 55(o_1 - 200))$. So you could modify the LP in part (a) by removing the constraint $0_1 \leq 200$, subtracting z from the objective, and adding new constraints $z \geq 0$ and $z \geq 55(o_1 - 200)$. (This can be thought of as minimising a maximum (rather than maximising a maximum) because z has a minus sign in the objective function that we are trying to maximise.)