Matrix - Linear transforation

8.1 Determinants of 2 x 2 matrix

Defn Let $A = (a_{ij})$ be a 2×2 matrix. The determinant of A, denoted $\det(A)$ is defined by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}.$$

$\det(A) = 0.$

$\det(A) = 2.$

$\det(A) = -1$

Thm 8.1.2 If A and B are 2×2 matrices then

(a) $\det(AB) = \det(A)\det(B)$

(b) $\det(A) \neq 0$ iff A is invertible

(c) if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is invertible then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

$\det(A) = 0$

$\rightarrow \infty$

Defn 8.1.3 If $A = (a_{ij})$ is a 3×3 matrix its determinant $\det(A)$ is defined by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \qquad (8.1.3)$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

$$- a_{21}(a_{12}a_{33} - a_{13} - a_{32})$$

$$+ a_{31}(a_{12}a_{23} - a_{13}a_{22}).$$

Example 8.2.1 if

$$A = \begin{pmatrix} 3 & 2 & 5 & -1 \\ -2 & 9 & 0 & 6 \\ 7 & -2 & -3 & 1 \\ 4 & -5 & 8 & -4 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 3 & 2 & -1 \\ 7 & -2 & 1 \\ 4 & -5 & -4 \end{pmatrix}$$

If we define the determinant of 1×1 matrix $A = (a_{ij})$ by

$$\det(A) = a_{11}.$$

if $A = (a_{ij})_{2\times 2}$ then

$$\det(A) = a_{11}\,\underline{\det(A_{11})} - a_{21}\,\underline{\det(A_{21})};$$

<u>if $A = (a_{ij})_{3\times 3}$</u>

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31}).$$

$$\det(A) = a_{11}\,a_{22} - a_{21}\,a_{12}$$

$$\det(A_{11}) = a_{22} \qquad \det(A_{21}) = a_{12}$$

Defn 8.2.2 Let $A = (a_{ij})$ be $n \times n$ matrix.
The determinant of A, $\det(A)$ is defined as follows:

- if $n = 1$, then $\det(A) = a_{11}$.
- if $n > 1$, then $\det(A)$ is the sum of n terms of the form $\pm a_{i1}\det(A_{i1})$ with plus and minus signs, and where the entries $a_{11}, a_{21}, \dots, a_{n1}$ are from the first column of A.

$$\boxed{\begin{aligned}\det(A) &= a_{11}\det(A_{11}) - a_{21}(A_{21}) + \dots + (-1)^{n+1} a_{n1}\det(A_{n1}) \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1}\det(A_{i1}).\end{aligned}}$$

Defn 8.2.4 A matrix $A = (a_{ij})$

(i,j) - cofactor of A is the ~~number~~

C_{ij} defined by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

the definition of $\det(A)$ reads

$$\det(A) = a_{11} C_{11} + a_{21} C_{21} + \dots + a_{n1} C_{n1}.$$

$\Rightarrow$ cofactor expansion down the first column of A.

Thm 8.2.5 (Cofactor Expansion Theorem)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any column or rows. The expansion down the j-th column is

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}.$$

and the cofactor expansion across the i-th row is

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}.$$

- Example.

$$A = \begin{pmatrix} 0 & 0 & 7 & -5 \\ -2 & 9 & 6 & -8 \\ 0 & 0 & -3 & 2 \\ 0 & 3 & -1 & 4 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 0 & 0 & 7 & -5 \\ -2 & 9 & 6 & -8 \\ 0 & 0 & -3 & 2 \\ 0 & 3 & 1 & 4 \end{vmatrix} = -(-2) \begin{vmatrix} 0 & 7 & -5 \\ 0 & -3 & 2 \\ 3 & -1 & 4 \end{vmatrix}$$

$$0 \begin{vmatrix} 9 & 6 & -8 \\ 0 & -3 & 2 \\ 3 & 1 & 4 \end{vmatrix} \qquad = 2 \cdot 3 \begin{vmatrix} 7 & -5 \\ -3 & 2 \end{vmatrix}$$

$$= 2 \cdot 3 \left(7 \cdot 2 - ((-3) \cdot (-5))\right)$$

$$0 \begin{vmatrix} 0 & 7 & -5 \\ 9 & 6 & -8 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= -6.$$

$$0$$

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{pmatrix}$$

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= (-1)^{2+1} a_{21} \det(A_{21})$$

$$+ (-1)^{2+2} a_{22} \det(A_{22})$$

$$+ (-1)^{2+3} a_{23} \det(A_{23}).$$

$$= -0 \cancel{\begin{vmatrix} -1 & 3 \\ 0 & 7 \end{vmatrix}} + 0 \cancel{\begin{vmatrix} 4 & 3 \\ 1 & 7 \end{vmatrix}}$$

$$-2 \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} = -2$$

$$-2\left(4 \cdot 0 - (1 \cdot (-1)\right).$$

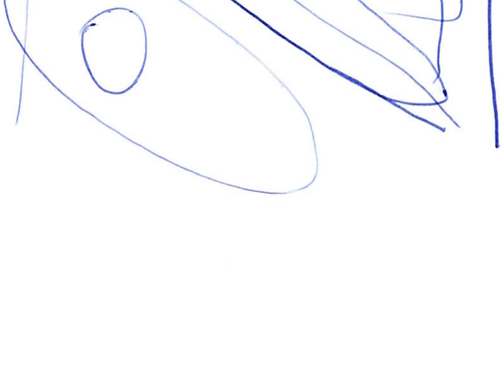
Example

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 & 0 \\ 9 & -6 & 4 & -1 & 3 \\ 2 & 4 & 0 & 0 & 2 \\ 8 & 3 & 1 & 0 & 7 \end{pmatrix} \qquad \det(A) = ?$$

$$= 3 \begin{vmatrix} 5 & 0 & 0 & 0 \\ -6 & 4 & -1 & 3 \\ 4 & 0 & 0 & 2 \\ 3 & 1 & 0 & 7 \end{vmatrix}$$

$$= 3 \cdot 5 \begin{vmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{vmatrix}$$

$$= 3 \cdot 5(-2) \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} = -30.$$

$A =$  $\det(A) =$

Theorem 8.2.8 If A is either an upper or a lowe triangular matrix, then $\det(A)$ is the product of the diagonal entries of A.

8.3 Properties of determinants

Theorem 8.3.1 Let A be a square matrix.

(a) If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.

(b) If one row of A is multiplied by α to produce B, then $\underline{\det(B) = \alpha \det(A)}$.

(c) If a multiple of one row of A is added to another row to produce a matrix B then $\det(B) = \det(A)$.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 1 & 2 \\ 3 & 12 & 9 \\ 1 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 2 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 9 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 7 & 3 & 9 \\ 0 & -2 & 1 \end{vmatrix} = B$

$\Downarrow \qquad \Downarrow$

$\Rightarrow \det(B) = \det(A)$

<u>Example</u>

$$\begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

$$= R_3 + 2R_2 \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

$$= R_3 - R_2 \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 9 \end{vmatrix}$$

$$= 0$$

Example 8.3.4 Compute det(A)

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \underline{R_2 + R_4}$$

$$= \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2$$

$$= -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= 3R_1 - R_2 \; (-2) \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$\cancel{= (-2)(-1)} = -30.$$