$X(YZ)=(XY)Z$. Thus we can

- ~~omit~~ omit the brackets and simply write XYZ.

Similarly for products of more than three matrices, if we don't change the order of the matrices. We don't the bracket.

- If A is a square matrix $A^k = \underbrace{A \cdot A \cdots A}_{k}$ k-th power of A.

- In general $AB \neq BA$. even if AB and BA have the same size

$A_{m\times n}$
$B_{n \times p}$ $\Rightarrow$ AB has the size $m \times p$

BA,
- if $p \neq m$, then we can multiply BA [sic]
- if $p = m$ we can do multiplication
  BA has the size $n \times n$
  AB - - - - $m \times m$
- if $p = m = n$,

$(AB)_{11}$

$$\begin{pmatrix} \cancel{a_{11} \cdots a_{1m}} \\ a_{21} \cdots a_{2m} \\ \vdots \\ a_{m1} \cdots a_{mm} \end{pmatrix} \begin{pmatrix} b_{11} \cdots b_{1m} \\ \vdots \\ b_{m1} \cdots b_{mm} \end{pmatrix}$$

AB had ij entrie $\sum_{k=1}^{m} a_{ik} \cdot b_{kj}$

$$\begin{pmatrix} \cancel{b_{11} \cdots b_{1m}} \\ \vdots \\ b_{m1} \cdots b_{mm} \end{pmatrix} \begin{pmatrix} a_{11} \cdots a_{1m} \\ \vdots \\ a_{m1} \cdots a_{mm} \end{pmatrix}$$

BA - - - - - - $\sum_{k=1}^{m} b_{ik} \cdot a_{kj}$

$(BA)_{11}$

Example (7.1.15)

$$A=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$AB=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$BA=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

- Definition. 7.1.16. If A and B are two matrices with $AB=BA$, then A and B are commute.
- Definition 7.1.17. If A is a square matrix, a matrix B is called an inverse of A if

$$AB=I \quad \text{and } BA=I.$$

, when I is the indentity matix with the same size of A, B.

A matrix that has an inverse, is called invertible.

- Example $\quad A=\begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad B=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}$

if A has an inverse B. then

B must have the size as 2×2.

$$AB=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}=\begin{pmatrix}b_{11}&b_{12}\\0&0\end{pmatrix}\neq I_2$$

Thus, A is not invertible.

- If a matrix is invertbibe then its inverse is unique.

Theorem 7.1.18. If B and C are both inverses of A, then $B = C$

Prof. Since B and C are inverses of A, we have $AB = I$ and $CA = I$, Thus.

$$B = IB = (CA)B = CAB = C(AB) = CI = C$$

If A is an invertible matix, the unique inverse of A is denoted by A^{-1},

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

Note that the above equality implies that if A is invertible, then its inverse A^{-1} is also invertible and

$$(A^{-1})^{-1} = A$$

Theoren 7.1.19 If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. observe that

$$(AB)\underline{\underline{(B^{-1}A^{-1})}} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$\underline{\underline{(B^{-1}A^{-1})}}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Thus $B^{-1}A^{-1}$ is the inverse of AB