**MTH5114** **Linear Programming and Games, Spring 2024**

**Week 8 Seminar Questions** **Viresh Patel**

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**Practice and Discussion Questions:**

1. Give the dual of the following linear program:

$$\begin{aligned}
\text{minimize} \quad & 3x_1 + 7x_3 + 9x_4 \\
\text{subject to} \quad & -x_1 + 3x_2 - x_3 - 10x_4 \leq -12, \\
& 2x_1 + 5x_2 + 8x_3 + 9x_4 \geq 1, \\
& x_1 + 4x_2 + 8x_3 + x_4 \geq 0, \\
& x_1, x_2 \geq 0, \\
& x_3 \leq 0, \\
& x_4 \text{ unrestricted}
\end{aligned}$$

**Solution:** First, we need to fix the constraints and variables in this program. We get:

$$\begin{aligned}
\text{minimize} \quad & 3x_1 - 7\bar{x}_3 + 9x_4 \\
\text{subject to} \quad & x_1 - 3x_2 - \bar{x}_3 + 10x_4 \geq 12, \\
& 2x_1 + 5x_2 - 8\bar{x}_3 + 9x_4 \geq 1, \\
& x_1 + 4x_2 - 8\bar{x}_3 + x_4 \geq 0, \\
& x_1, x_2 \geq 0, \\
& \bar{x}_3 \geq 0, \\
& x_4 \text{ unrestricted}
\end{aligned}$$

(Notice that when fixing the constraints here, we have to make all constraints with "$\geq$" because our starting LP is a minimisation.)

Then, taking the dual gives:

$$\begin{aligned}
\text{maximize} \quad & 12y_1 + y_2 \\
\text{subject to} \quad & y_1 + 2y_2 + y_3 \leq 3, \\
& -3y_1 + 5y_2 + 4y_3 \leq 0, \\
& -y_1 - 8y_2 - 8y_3 \leq -7, \\
& 10y_1 + 9y_2 + y_3 = 9, \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

2. Show that if a linear program is unbounded, its dual must be infeasible.

   **Solution:** Consider a linear program with objective function $\mathbf{c}^\mathsf{T}\mathbf{x}$, and let its dual have objective function $\mathbf{b}^\mathsf{T}\mathbf{y}$. We prove the contrapositive of the statement i.e. that if the dual LP is *not* infeasible, then the primal LP is *not* be unbounded.

   Suppose the dual has some feasible solution $\mathbf{y}$, and let $k = \mathbf{b}^\mathsf{T}\mathbf{y}$. Then, we know from the Weak Duality Theorem that $\mathbf{c}^\mathsf{T}\mathbf{x} \leq k$ for any feasible solution $\mathbf{x}$ of the primal LP. This shows that the primal LP cannot be unbounded by the definition of what it means to be unbounded. (Recall the definition of unboundedness for an LP: the primal LP is unbounded if, for every $k \geq 0$, there is a feasible solution $\mathbf{x}$ such that $\mathbf{c}^\mathsf{T}\mathbf{x} \leq k$.)

3. Can a linear program and its dual both be unbounded? Can both be infeasible?

   **Solution:** Both cannot be unbounded, because we have just shown that if one is unbounded, then other must in fact have *no* feasible solutions, and the definition of unbounded requires that we can find a feasible solution at least as large as any given value $k$.

   It turns out that both *can* be infeasible. Consider the following:

$$\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leq -1, \\
& -x_1 + x_2 \geq 1, \\
& x_1, x_2 \geq 0
\end{aligned}$$

   Obviously this program is infeasible, since we can't find any value for $-x_1 + x_2$ that is both at most -1 and at least 1. Let's take the dual. Rewriting the program gives:

$$\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1 - x_2 \leq -1, \\
& -x_1 + x_2 \leq -1, \\
& x_1, x_2 \geq 0
\end{aligned}$$

   Then, taking the dual we obtain:

$$\begin{aligned}
\text{minimize} \quad & -y_1 - y_2 \\
\text{subject to} \quad & y_1 - y_2 \geq 1, \\
& -y_1 + y_2 \geq 1, \\
& y_1, y_2 \geq 0
\end{aligned}$$

Rearranging gives us:

$$\begin{aligned} \text{minimize} \quad & -y_1 - y_2 \\ \text{subject to} \quad & y_1 - y_2 \geq 1, \\ & y_1 - y_2 \leq -1, \\ & y_1, y_2 \geq 0 \end{aligned}$$

Here, we see the same problem! There is no way to make $y_1 - y_2$ at least 1 and at most $-1$ at the same time.

4. In lecture, we saw how to find the dual of an LP that is not in standard inequality form. We saw that equations in the primal LP became unrestricted variables in the dual LP (and vice versa). What if instead, we converted the primal LP to standard inequality form and then took the dual? Is this consistent?

   **Solution:** Let's think first about an unrestricted variables $x$ in the program. When we convert to standard inequality form, this will be replaced by $x^+ - x^-$. This takes the column for $x$ and replaces it by 2 columns: one for $x^+$ with the same coefficients as $x$, and one for $x^-$ in which these coefficients have all been negated. When we take the dual, we get now 2 dual constraints for these columns:

$$\begin{aligned} a_1 y_1 + a_2 y_2 + \cdots + a_n y_n &\geq b \\ -a_1 y_1 - a_2 y_2 - \cdots - a_n y_n &\geq -b \end{aligned}$$

   where $a_1, a_2, \cdots, a_n$ are the entries in the column for our original variable $x$ in the original program. Notice this is exactly like adding an equation:

$$a_1 x_1^+ + a_2 x_2^+ + \cdots + a_n x_n^+ = b$$

   Similarly, consider what happens to an equation in the primal, of the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.$$

   When we convert to standard inequality form we "split" this into a pair of inequalities:

$$\begin{aligned} a_1 x_1 + a_2 x_2 + \cdots + a_n x_n &\leq b. \\ -a_1 x_1 - a_2 x_2 - \cdots - a_n x_n &\leq -b. \end{aligned}$$

   When we take the dual, we now get 2 variables, one for each inequality. Call the first one $y^+$ and the second one $y^-$. Notice that in the $i$th dual constraint, the coefficient of $y^+$ will be $a_i$ and the coefficient of $y^-$ will be $-a_i$. Thus, we get $a_i(y^+ - y^-)$ in the $i$th dual constraint. But this is exactly like having a single unrestricted variable $y$ that is multiplied by $a_i$ in the $i$th constraint.