MTH6158 Ring Theory: Guide to Coursework 2

Note: This guide is meant to help you understand and carry out the problem solutions on your own. It is **not** meant to provide complete solutions!

1. *Consider the ring* $R = \mathbb{Z}/12\mathbb{Z} = \{[0]_{12}, [1]_{12}, \ldots, [11]_{12}\}$*, and let* I *be its ideal* $I = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$.

   (a) *List explicitly all the cosets of* I *in* R*.*

   The ring R has 12 elements and the ideal I has 4 elements, so the partition of R into cosets should consist of 3 cosets of size 4 each. One of them is always I, and all the others should have the form $I + r$ for some r in R. Following this reasoning, you should get that the three cosets are

   $$A := \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$$
   $$B := \{[1]_{12}, [4]_{12}, [7]_{12}, [10]_{12}\}$$
   $$C := \{[2]_{12}, [5]_{12}, [8]_{12}, [11]_{12}\}$$

   (b) *Write down the addition and multiplication tables for* R/I*.*

   Addition and multiplication of cosets are defined via their representatives. You should check that this leads to the following addition and multiplication tables:

+	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

·	A	B	C
A	A	A	A
B	A	B	C
C	A	C	B

   (c) *Prove that* $R/I \cong \mathbb{Z}/3\mathbb{Z}$*, by giving an explicit isomorphism (there is no need to prove formally that it is an isomorphism).*

   The ring $\mathbb{Z}/3\mathbb{Z}$ has 3 elements $[0]_3, [1]_3, [2]_3$, which we know how to add and multiply together. Looking at the addition and multiplication tables for R/I above, you should be able to see which coset plays the role of $[0]_3$, which one plays the role of $[1]_3$, and which one plays the role of $[2]_3$. With this, you should be able to define a concrete isomorphism:

   $$\begin{aligned} R/I &\longrightarrow \mathbb{Z}/3\mathbb{Z} \\ A &\longmapsto ? \\ B &\longmapsto ? \\ C &\longmapsto ? \end{aligned}$$

2. *Suppose that R_1 is a ring with addition $+_1$ and multiplication $\cdot_1$, and that R_2 is a ring with addition $+_2$ and multiplication $\cdot_2$. Prove that the set $R_1 \times R_2$, with addition given by $(r_1, r_2)+(s_1, s_2) := (r_1 +_1 s_1, r_2 +_2 s_2)$ and multiplication given by $(r_1, r_2) \cdot (s_1, s_2) := (r_1 \cdot_1 s_1, r_2 \cdot_2 s_2)$ is a ring. This ring is called the* product *of rings R_1 and R_2.*

   You need to show that $R_1 \times R_2$ satisfies all the axioms of a ring. This follows quite directly from the fact that both R_1 and R_2 satisfy these axioms – make sure to say explicitly, for example, what the zero of $R_1 \times R_2$ is, or what the negative of an element (r_1, r_2) is.

3. *Denote by $\mathbb{Z}_m = \{[0]_m, [1]_m, \ldots, [m-1]_m\}$ the ring of integers modulo m. Consider the rings $R = \mathbb{Z}_{24}$ and $S = \mathbb{Z}_4 \times \mathbb{Z}_6$. Let $\theta : R \to S$ be the map defined by $\theta([x]_{24}) = ([x]_4, [4x]_6)$.*

   (a) *Prove that θ is well-defined, that is, does not depend on the choice of coset representative.*
   Here you need to show that for any two integers $x, y \in \mathbb{Z}$, if $[x]_{24} = [y]_{24}$ then $[x]_4 = [y]_4$ and $[4x]_6 = [4y]_6$. For this, simply use the definition of when two integers are in the same equivalence class of $\mathbb{Z}_m$.

   (b) *Is θ a homomorphism of rings? Explain.*
   After unpacking the definition, this question reduces to seeing if

   $$(\,[x+y]_4\,,\,[4(x+y)]_6\,) \stackrel{?}{=} ([x]_4, [4x]_6) \,+\, ([y]_4, [4y]_6)$$

   and

   $$(\,[x \cdot y]_4\,,\,[4(x \cdot y)]_6\,) \stackrel{?}{=} ([x]_4, [4x]_6) \,\cdot\, ([y]_4, [4y]_6)\,.$$

   Both of these are true, so θ is indeed a ring homomorphism. Be careful when proving the second of these equalities, as it involves a non-trivial step.

   (c) *Is θ an isomorphism of rings? Explain.*
   Since θ is a homomorphism, the question is basically asking if θ is a bijection. Despite R and S having the same number of elements, this is not the case, as θ is neither injective nor surjective – make sure to explain why.

   (d) *What are the image and the kernel of θ?*
   The kernel of θ is equal to $\{[0]_{24}, [12]_{24}\}$. The image of θ is equal to $\{([x]_4, [y]_6) \mid y \text{ is even}\}$. Can you see why?

   (e) *What does the First Isomorphism Theorem say in this case? Write down the explicit isomorphism provided by this theorem.*
   It says that

   $$\mathbb{Z}_{24} \Big/ \{[0]_{24}, [12]_{24}\} \cong \{([x]_4, [y]_6) \mid y \text{ is even}\}.$$

   The isomorphism sends the coset of $\mathrm{Ker}(\theta)$ containing an element $[x]_{24}$ to the element $([x]_4, [4x]_6)$ of $\mathrm{Im}(\theta)$.

4. *Consider the ring $R = 2\mathbb{Z}$ and the ideal $I = 24\mathbb{Z}$ of R.*

   (a) *Give a representative for each coset of I in R.*
   There are 12 cosets of I in R, with representatives

$$0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22 \in R.$$

   (b) *Does the ring R/I have an identity? Explain.*
   R/I does not have an identity. To argue why, it is not enough to say that R does not have an identity, as it could still be the case that the quotient ring R/I does. But in this particular case, you can check that none of the 12 elements of R/I is a multiplicative identity – for instance, no element x of R/I satisfies $x \cdot x = x$, which a multiplicative identity should.

   (c) *Use the Second Isomorphism Theorem to list all the subrings of R/I.*
   By the Second Isomorphism Theorem, subrings of R/I are in correspondence with subrings of R containing I. These are subrings of the form $m\mathbb{Z}$ with $2 \mid m \mid 24$. There are six such subrings of R, which are $2\mathbb{Z}$, $4\mathbb{Z}$, $6\mathbb{Z}$, $8\mathbb{Z}$, $12\mathbb{Z}$, and $24\mathbb{Z}$. Make sure you can describe explicitly the six corresponding subrings of R/I!

5. *Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider the ring $R[x]$ of polynomials with coefficients in R.*

   (a) Is $R[x]$ a domain? Is $R[x]$ an integral domain? Explain.
   Since R is a domain, $R[x]$ is also a domain (a commutative ring with identity). For example, what is the identity of $R[x]$? However, the ring R has zero-divisors, and this leads to many zero-divisors in $R[x]$. For example, $([2]_4\, x) \cdot ([2]_4 + [2]_4\, x) = 0$ (the zero polynomial), but none of the factors is the zero polynomial. This means that $R[x]$ is not an integral domain.

   (b) *Is the element $[2]_4 + [0]_4\, x + [2]_4\, x^2 \in R[x]$ a zero-divisor?*
   It is. Can you find another non-zero polynomial which multiplied by it is equal to 0?

   (c) *Is the element $[2]_4 + [1]_4\, x \in R[x]$ a zero-divisor?*
   It is not. To see this, think about what the leading coefficient (the coefficient with the highest power of x) is after multiplying by another polynomial.

   (d) *Is the element $[2]_4 + [3]_4\, x + [2]_4\, x^2 \in R[x]$ a unit?*
   It is not. Think about the constant coefficient after multiplying by another polynomial.

   (e) *Is the element $[3]_4 + [2]_4\, x \in R[x]$ a unit?*
   It is! Can you find its inverse?