Inverse of a matrix

Claudia Garetto

Queen Mary University of London

c.garetto@qmul.ac.uk

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Why to invert a matrix

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

$$A = (a_{ij})$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} n$$
$$n$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

A is invertible

$$A^{-1}A = I$$
$$AA^{-1} = I$$

$$\underset{n\times n}{A}\,\underset{n\times 1}{\mathbf{x}} = \underset{n\times 1}{b}$$
$$A^{-1}(Ax) = \underset{n\times n}{A^{-1}}\,\underset{n\times 1}{b}$$
$$(A^{-1}A)x = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

$x \in \mathbb{R}$

$2x = 3$

$\boxed{ax = b}$ $a, b \in \mathbb{R}$

$a \neq 0$

$$\frac{1}{a}\, a x = \frac{1}{a} \cdot b$$
$$1 \cdot x = \frac{b}{a}$$
$$x = \frac{b}{a}$$

Lemma

Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Suppose that M is an invertible $m \times m$ matrix. The following two systems are equivalent (i.e. they have the same set of solutions):

M invertible

$$(1) \qquad A\mathbf{x} = \mathbf{b}$$

$$(2) \qquad MA\mathbf{x} = M\mathbf{b}$$

Proof $\quad x$ solves $(1) \iff x$ solves (2)

$\Longrightarrow$

$\Longleftarrow$

$\Longrightarrow$ If x solves (1) then $Ax = b$

$$MAx = M(Ax) \overset{(1)}{=} Mb \qquad x \text{ solves } (2)$$

$\Longleftarrow$ If x solves (2) $\quad MAx = Mb$. Multiply both sides by M^{-1}

$$M^{-1}(MAx) = M^{-1}(Mb)$$
$$(M^{-1}M)Ax = (M^{-1}M)b$$
$$Ax = b$$

Definition

An **elementary matrix** of **type I** (respectively, **type II**, **type III**) is a matrix obtained by applying an elementary row operation of type I (respectively, type II, type III) to an identity matrix.

$I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Elementary matrix of type I

$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad R_1 \Leftrightarrow R_2$

Elementary matrix of type II

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Elementary matrix of type III

$E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad R_1 \to R_1 + 2R_3$

Elementary row operations

Type I	interchanging two rows;	$R_i \Leftrightarrow R_j$
Type II	multiplying a row by a non-zero scalar; $\alpha \neq 0$	αR_i
Type III	adding a multiple of one row to another row.	$R_i + \alpha R_j \quad \alpha \in \mathbb{R}$

Examples

Type I $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Type II $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Type III $E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A = (a_{ij})_{i,j=1,\dots,3}$

E_1: I swapped the first row in I with the second

$$E_1 A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

E_2: Multiply the third row in I by 4

$$E_2 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{pmatrix}$$

E_3: $R_1 \to R_1 + 2R_3$

$$E_3 A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + 2a_{31} & a_{12} + 2a_{32} & a_{13} + 2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Theorem

If E is an $m \times m$ elementary matrix obtained from I by an elementary row operation, then left-multiplying an $m \times m$ matrix A by E has the effect of performing that same row operation on A.

$E_1 A$

$E_2 A$

$E_3 A$

Theorem

If E is an elementary matrix, then E is invertible and E^{-1} is an elementary matrix of the same type.

<u>Proof</u> We need to prove that E is invertible and that E^{-1} is an elementary matrix of the same type

~~Obser~~ <u>Observation</u> :

- if two rows of a matrix are interchanged, then interchanging them again restores the original matrix
- If a row is multiplied by $\alpha \neq 0$, then multiplying this row by $\frac{1}{\alpha}$ restores the original matrix.
- if α times row q has been added to row r, then adding $-\alpha$ times row q to row r restores the original matrix.

Let E be an elementary matrix. (E is obtained from I via a row operation of type I, II or III)

There exists a row operation that brings E back to I: $\exists F$ elementary matrix of the same type such that

$$FE = I$$

In addition

$$EF = I$$

$\Longrightarrow$ E invertible, $E^{-1} = F$ is an elementary matrix of the same type $\square$

Example

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~Revers~~ Interchanging row 1 with row 2 in the identity matrix

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

I multiply the third row by 4

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 \longrightarrow R_1 + 2R_3$$

Elementary matrix of type III with $R_1 \to R_1 - 2R_3$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition

A matrix B is **row equivalent** to a matrix A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A\,.$$

$A = ?$

$$B = E_k E_{k-1} \cdots E_1 A$$

$$E_k^{-1} B = (E_k^{-1} E_k) E_{k-1} \cdots E_1 A$$

$$E_k^{-1} B = E_{k-1} E_{k-2} \cdots E_1 A$$

$$E_{k-1}^{-1} E_k^{-1} B = E_{k-2} \cdots E_1 A$$

$$\boxed{E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B = A}$$

(a) A is row equivalent to itself;

(b) if A is row equivalent to B, then B is row equivalent to A;

(c) if A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Proof

(c) $\Longrightarrow$ (d) by hypothesis

$$A = E_k E_{k-1} \cdots E_2 E_1 I = E_k E_{k-1} \cdots E_2 E_1$$

(d) $\Longrightarrow$ (a) Let $A = E_k E_{k-1} \cdots E_2 E_1$ elementary matrices

A is a product of invertible matrices and therefore it is invertible

By the theory of invertible matrices we have that

$$A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$$

$\boxtimes$

Invertible matrix theorem

Theorem

Let A be a square $n \times n$ matrix. The following are equivalent:

(a) A is invertible;

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution; $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

(c) A is row equivalent to I;

(d) A is a product of elementary matrices.

$$(a) \Downarrow (b) \Downarrow (c) \Downarrow (d) \Longrightarrow (a)$$

<u>Proof</u> (a) $\Rightarrow$ (b) I know that A is invertible $\left(\exists A^{-1}: \begin{matrix} A^{-1}A = I \\ AA^{-1} = I \end{matrix}\right)$

$$Ax = 0$$
$$A^{-1}(Ax) = A^{-1} \cdot 0$$
$$(A^{-1}A)x = A^{-1} \cdot 0$$
$$x = A^{-1} \cdot 0 = 0$$

(b) $\Rightarrow$ (c) $\quad Ax = 0 \Longrightarrow x = 0$

Use elementary row operations to bring the system $Ax = 0$ to the equivalent form $Ux = 0$ where U is in row echelon form.

by our assumption we have that $Ux = 0$ has $x = 0$ ~~as~~ unique solution.

In this method we have used elementary row transformations that show that A is row equivalent to I.

Corollary

Suppose that A and C are square matrices such that $\boxed{CA = I.}$ Then also $AC = I$; in particular, both A and C are invertible with $C = A^{-1}$ and $A = C^{-1}$.

Proof

I want to prove that A is invertible and that $A^{-1} = C$.

By the invertible matrix theorem I can equivalently prove that $Ax = 0$ implies $x = 0$.

$\boxed{Ax = 0}$ $\quad x = I \cdot x = CAx = C(Ax) = C \cdot 0 = 0$

I have proven that A is invertible $\left(\exists A^{-1} \quad A^{-1}A = I, \ AA^{-1} = I\right)$

I want to prove that $A^{-1} = C$

$$C = C\underline{I} = CAA^{-1} = (CA)A^{-1} = IA^{-1} = A^{-1} \qquad \square$$

Examples