Vectors & Matrices

Problem Sheet 8

1. Consider the planes Π_1, Π_2, Π_3 defined by the Cartesian equations

$$\begin{aligned} 3x - y + 7z &= 8 \ , \\ -2x - 3y + 5z &= 3 \ , \\ x + y - 2z &= -2 \ . \end{aligned}$$

   (i) Express the linear system formed by these three equations as a single matrix equation, $A\mathbf{x} = \mathbf{b}$.

   (ii) Let B be the 3×3 matrix given by

$$\begin{pmatrix} 4 & 0 & 5 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} .$$

   Show that there exists two Type III elementary matrices E_1, E_2 such that $B = E_1 E_2 A$.

   (iii) Find three values $c_1, c_2, c_3 \in \mathbb{R}$ such that the planes defined by the Cartesian equations

$$\begin{aligned} 4x + 5z &= c_1 \ , \\ -y + z &= c_2 \ , \\ x + y - 2z &= c_3 \ , \end{aligned}$$

   have the exact same intersection as the planes Π_1, Π_2, Π_3.

2. A Type I elementary matrix E is a matrix such that the effect of left-multiplying A by E is to swap around a pair of rows inside A, i.e. EA is the same matrix as A, but with two rows swapped.

   (i) List **all** possible 4×4 Type I elementary matrices.

   (ii) How many $n \times n$ Type I elementary matrices exist?

   (iii) How many $n \times n$ matrices exist that can be expressed as the product of Type I elementary matrices? (**Note:** There can be any number of Type I factors, not necessarily just two.)

   (iv) List **all** possible 3×3 matrices that can be expressed as the product of Type I elementary matrices.

3. Suppose we represent quadratic functions $q(x) = ax^2 + bx + c$ by the column vector in 3-space containing their coefficients,

$$\mathbf{q} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 .$$

(i) Find two elementary matrices E_1 and E_2 such that left-multiplying the vector $\mathbf{q}$ by the product E_1E_2 gives a vector $E_1E_2\,\mathbf{q}$ representing the quadratic $q(-3x)$.

(ii) Suppose linear functions are similarly represented by vectors in $\mathbb{R}^2$. Find a matrix M that maps a vector representing a linear function $f(x)$, to one representing the quadratic $(2x - 3)f(x)$.

(iii) Find a matrix D that maps a vector representing a quadratic $q(x)$ to the vector representing its derivative $q'(x)$.

(iv) Let N be a map that takes a linear function $f(x)$ and gives the anti-derivative $F(x)$ satisfying the condition $F(-1) = 0$. Express this map N as a matrix.

(v) Show that $DN = I_2$. Is the same true for ND?