Mathematical Tools for Asset Management MTH6113

Factor Models and Arbitrage Pricing Theory (APT)

Dr. Melania Nica

Reminder: Capital Asset Pricing Model (CAPM)I

CAPM: relation between **expected return on any asset** i and the return on the market

$$E\left(R_i\right) = r + \left(E_M - r\right)\beta_i$$

$E_M - r$: **Market Risk Premium**

$\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)}$

Reminder: Capital Asset Pricing Model (CAPM)

- Market portfolio not observable
- Empirical test use broad-based equity index such FTSE-100, S&P500, Nikkei 250 as proxies

$$R_i = r + \beta_i \left(R_{index} - r \right) + \varepsilon_i$$

  - empirical studies do not strongly support the model
- The true market portfolio might contain other financial assets such as bonds and stock that are not in included in this indices as well as non financial assets: (real-estate, human capital).

Factor Models: Single Factor Models

Returns on stock i depend on a common factor I which affects all assets:

$$R_i = \alpha_i + \beta_i I + \varepsilon_i$$

with

$$\begin{aligned} E(\varepsilon_i) &= 0 \text{ for any } i \\ Cov(\varepsilon_i, \varepsilon_j) &= 0 \text{ for any } i, j \\ Cov(\varepsilon_i, I) &= 0 \text{ for any } i \end{aligned}$$

α_i : asset specific constant
β_i : asset specific factor loading or factor sensitivity
ε_i : error term uncorrelated across states
I : factor which affects all assets with different sensitivity

Single Factor Models

How we estimate α_i and β_i?

- We need some data for the random variables R_i and I.
- Suitable data might be daily or monthly or annual returns.
- **Ordinary Least Squares estimator**:
  - minimise the residual sum of squares: $\min_{\alpha,\beta} \sum_\iota \varepsilon_i^2$
- $\hat{\beta}_{OLS} = \frac{Cov(R_i, I)}{Var(I)}$

CAPM as a Single Factor Model

Accounting for an error terms we could rewrite CAPM as:

$$R_i - r = \beta_i \left(R_M - r \right) + \varepsilon_i$$

- Test: OLS regression without intercept using historical data for R_i and R_M
- $\hat{\beta}_{OLS} = \frac{Cov(R_i, R_M)}{Var(R_M)}$ beta estimate based on historical data.

Multifactor Models

- Assume returns are influenced not only by *market movements* but also by *other factors*
  - systematic risks (which cannot be diversified away)
- Multifactor Models:

$$R_i = \alpha_i + \beta_{i,1} I_1 + \beta_{i,2} I_2 + ... + \beta_{i,L} I_L + \varepsilon_i$$

$$\begin{aligned} E\left(\varepsilon_i\right) &= 0 \text{ for any } i \\ Cov\left(\varepsilon_i, \varepsilon_j\right) &= 0 \text{ for any } i, j \\ Cov\left(\varepsilon_i, I_l\right) &= 0 \text{ for any } i, l \\ E\left(I_l\right) &= 0 \text{ for any } l \end{aligned}$$

Multifactor Models

- I_l random variable
- I_l factors capture the variation of R_i about the expected return
- $E(R_i) = \alpha_i$
- Factors should proxy for risks and may be identified from economic fundamentals or empirical observations

Multifactor Models

Macroeconomic Factor Models

- The systematic forces influencing returns must be those affecting discount factors and expected cash flows.
- A set of candidates such as
  - industrial production
  - expected and unexpected inflation
  - measures of the risk premium
  - the term structure
  - oil prices.

- Empirically:
  - industrial production (affecting cash flow expectations)
  - changes in the risk premium measured as the spread between the yields on low-and high-risk corporate bonds (witnessing changes in the market risk appetite),
  - twists in the yield curve, as measured by the spread between short and long term interest rates (representing movements in the market rate of impatience).
  - measures of unanticipated inflation and changes in expected inflation also play a (less important) role.

Models with factor-mimicking portfolios

- The size and value factors
- Fama and French (1993) construct two-factor portfolios : HML (high-minus-low) and SMB (small-minus-big).
  - Both portfolios consist of a joint long and short position and have net asset value zero.
  - The HML portfolio represents a combination of a long position in high book-to-market stocks with a short position in low book-to-market stocks.
  - The SMB portfolio is one consisting of a long position in small capitalization stocks and a short position in large capitalization stocks.
  - In conjunction with the excess (above the risk-free rate) return on a broad-based index, these factors have highly significant explanatory power for the cross-section of stock returns.

Models with factor-mimicking portfolios

- Momentum portfolios
  - At the start of each month stocks are ranked according to historical returns in the 3,6,9, and 12- month prior periods. They are then assigned to one of ten equally weighted portfolios ranked by returns . A "buy-sell" portfolio is constructed where the lowest decile portfolio is sold short to finance an equal-value long position in the highest decile portfolio. The return on this long -short porfolio for 3,6,9,12-month forward horizon is then computed. Each month the portfolios are reassembled as per above.

Multifactor Models

Moving from individual stocks to portfolios

- **A security** is a broad-based portfolio
  - we ignore the idiosyncratic risk as it is diversified away
  - the only risk that matters is the factor risk!!
- **Aim: write any security** as a multifactor model

Multifactor Models

Example: Economy in which only stocks X, Y, X exist.
The returns on stocks X, Y and Z are determined by the following two factor model:

$$
\begin{aligned}
r_X &= 0.06 + 0.02 I_2 + \varepsilon_X \\
r_Y &= 0.08 + 0.02 I_1 + 0.01 I_2 + \varepsilon_Y \\
r_Z &= 0.15 + 0.04 I_1 + 0.04 I_2 + \varepsilon_Z
\end{aligned}
$$

Aim: to write any security (portfolio made of these assets) as models of the same factors
Q: What are the expected returns of X, Y, Z?
A: $E(r_X) = 0.06$, $E(r_Y) = 0.08$, $E(r_Z) = 0.15$

Multifactor Models

Let's find the return of an equally weighted portfolio of assets X, Y and Z

The return on this portfolio is

$$r_P = \frac{1}{3}r_X + \frac{1}{3}r_Y + \frac{1}{3}r_Z$$

$$\begin{aligned} r_P &= \frac{1}{3}\left(0.06 + 0.08 + 0.15\right) + \frac{1}{3}\left(0.02 + 0.04\right) I_1 + \\ &+ \frac{1}{3}\left(0.02 + 0.01 + 0.04\right) I_2 + \varepsilon \\ r_p &= 0.097 + 0.020 I_1 + 0.023 I_2 + \varepsilon \end{aligned}$$

Portfolio sensitivity to factor I_1 is 0.020

Portfolio sensitivity to factor I_2 is 0.023

Return of this portfolio is 0.097

Factor Replicating Portfolios

Definition: **A factor-replicating portfolio or pure factor portfolio** is a portfolio with unit exposure to one factor and zero exposure to others

Example
Factor 1 **replicating portfolio** has a sensitivity to factor I_1 equal to 1 and to factor I_2 is equal to 0.

What are the portfolio weights on X, Y and Z (w_X, w_Y, w_Z) of this portfolio?

$$\begin{aligned} 0.02w_Y + 0.04w_Z &= 1 \\ 0.02w_X + 0.01w_Y + 0.04w_Z &= 0 \\ w_X + w_Y + w_Z &= 1 \end{aligned}$$

Factor Replicating Portfolios

Answer:

$$w_X = -37, w_Y = 26, w_Z = 12$$

The pure factor 1 portfolio is:

$$R_{I_1} = -37 \times 0.06 + 26 \times 0.08 + 12 \times 0.15 + I_1 + error$$

or

$$R_{I_1} = 1.66 + I_1 + error$$

Factor Replicating Portfolios

Factor 2 **replicating portfolio** has a sensitivity to factor I_1 equal to 0 and to factor I_2 is equal to 1.

What are the portfolio weights on X, Y and Z (m_X, m_Y, m_Z) of this portfolio?

$$
\begin{aligned}
0.02m_Y + 0.04m_Z &= 0 \\
0.02m_X + 0.01m_Y + 0.04m_Z &= 1 \\
m_X + m_Y + m_Z &= 1
\end{aligned}
$$

Answer

$$m_X = 25.5, w_Y = -49, w_Z = 24.5$$

The pure factor 2 portfolio:

$$R_{I_2} = 25.5 \times 0.06 - 49 \times 0.08 + 25.5 \times 0.15 + I_2 + error$$

or

$$R_{I_2} = 1.285 + I_2 + error$$

Factor Replicating Portfolios

Could I replicate a risk free asset?
Risk free asset replicating portfolio has a sensitivity to factor I_1 equal to 0 and to factor I_2 is equal to 0.

What are the portfolio weights on X, Y and Z (n_X, n_Y, n_Z) of this portfolio?

$$\begin{aligned} 0.02n_Y + 0.04n_Z &= 0 \\ 0.02n_X + 0.01n_Y + 0.04n_Z &= 0 \\ n_X + n_Y + n_Z &= 1 \end{aligned}$$

Answer

$$n_X = 0.5, w_Y = 1, w_Z = -0.5$$

Thus,

$$R_f = 0.035 + error$$

Factor Replicating Portfolios

The return on pure factor 1 portfolio is

$$\begin{aligned} R_{I_1} &= 1.66 + I_1 + error \text{ thus} \\ E\left(R_{I_i}\right) &= 1.66 \end{aligned}$$

The return on pure factor 2 portfolio is

$$\begin{aligned} R_{I_2} &= 1.285 + I_2 + error \text{ thus} \\ E\left(R_{I_2}\right) &= 1.285 \end{aligned}$$

The return on an asset replicating the risk free asset is

$$\begin{aligned} R_f &= 0.035 + error \\ r &= E\left(R_f\right) = 0.035 \end{aligned}$$

Factor Risk Premiums

We denote factor risk premium: $\lambda_I = E\left(R_{I_I}\right) - r$
Remember in the CAPM: $E_M - r$ market risk premium
Thus:
Factor 1 risk premium is $1.66 - 0.035 = 1.625$
Factor 2 risk premium is $1.285 - 0.035 = 1.25$
Expected return on each factor is made of risk premium and risk free rate:

$$E\left(R_{I_I}\right) = \lambda_I + r$$

Tracking Portfolio

Asset X tracking portfolio or asset X replicating portfolio

- build a security or portfolio made of assets X, Z, Y which tracks exactly asset X
- sensitivity $\beta_{1X} = 0$ to the first factor
- sensitivity $\beta_{2X} = 0.02$ to the second factor?

We construct portfolio tracking X with:

- weight β_{1X} on the first factor replicating portfolio R_{I_1}
- weight β_{2X} on the second factor replicating portfolioR_{I_2}
- weight $1 - \beta_{1X} - \beta_{2X} = 0.98$ on the risk free asset.
- The expected return on the replicating (tracking) portfolio is:

$$
\begin{aligned}
E\left(R_X\right) &= \beta_{1X}\left(r+\lambda_1\right)+\beta_{2X}\left(r+\lambda_2\right)+\left(1-\beta_{1X}-\beta_{2X}\right)r \\
E\left(R_X\right) &= r+\beta_{1X}\lambda_1+\beta_{2X}\lambda_2
\end{aligned}
$$

- Remember the return on pure factor portfolio is $E\left(R_{I_l}\right) = r + \lambda_l$
- Substituting the values found earlier for λ_1, λ_2 and r we get $E\left(R_X\right) = 0.06$

Tracking Portfolios and Arbitrage Pricing Theory

Tracking Portfolios form the basis for Arbitrage Pricing Theory

- In the absence of arbitrage we require **all assets with identical factor exposures to earn the same return**
- **APT**: All securities and portfolios have expected returns described by:

$$E\left(R_i\right) = r + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + ... + \beta_{iL}\lambda_L$$

λ_l the risk associated with the lth factor

- **APT**: The expected return of any financial asset can be written as **the risk free rate plus sum of asset's factor sensitivities multiplied by the factor risk premiums which are invariant across the states**

$$E\left(R_i\right) = r + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + ... + \beta_{iL}\lambda_L$$

- If APT does not hold arbitrage opportunity exist

Arbitrage: generally describes risk-free profit.

- An arbitrage strategy: that delivers non-negative returns in all states of the world and strictly positive returns in at least one state of the world
  - faced with such strategy an investor will invest on an infinite scale
- financial markets do not permit however the existence of arbitrage strategies
- investors, irrespective of their initial wealth or risk aversion, are willing to take infinite positions in arbitrage opportunities and in turn driving prices to equilibrium.

Arbitrage Pricing Theory

Going back to the original example:
The returns on stocks X, Y and Z are determined by the following two factor model:

$$\begin{aligned} r_X &= 0.06 + 0.02 I_2 + \varepsilon_X \\ r_Y &= 0.08 + 0.02 I_1 + 0.01 I_2 + \varepsilon_Y \\ r_Z &= 0.15 + 0.04 I_1 + 0.04 I_2 + \varepsilon_Z \end{aligned}$$

Is there an arbitrage opportunity if the risk free rate in this economy is $r = 0.05$?

Arbitrage Pricing Theory

APT tells us that $E(R_X) = r + \beta_{1X}\lambda_1 + \beta_{2X}\lambda_2$, thus:

$$\begin{aligned} X &: \quad 0.06 = 0.05 + 0.02\lambda_2 \\ Y &: \quad 0.08 = 0.05 + 0.02\lambda_1 + 0.01\lambda_2 \\ Z &: \quad 0.15 = 0.05 + 0.04\lambda_1 + 0.04\lambda_2 \end{aligned}$$

Solving for λ_1 and λ_2 in the first two equations we get $\lambda_1 = 0.5$ and $\lambda_2 = 1.25$ and thus:

$$\begin{aligned} Z &: \quad 0.15 = 0.12 \\ & \quad \textit{ARBITRAGE!!} \end{aligned}$$

Long position in Z and an equal short position in its tracking portfolio formed from security X, Y and risk free asset

Arbitrage Pricing Theory

- APT has no particular role for the "Market Portfolio," which can't be measured anyway
- APT easily extended to multiple systematic factors
- APT employs fewer restrictive assumptions
- APT: based on no-arbitrage condition
- very general (also a weakness)

Arbitrage Pricing Theory

- Helps identify the sources of risk/split the systematic risks into more detailed components and thus permits modulating one's risk exposure
- E.g. two stocks with same beta could have different sensitivities to specific risk factors; useful in managing risk exposure or helping refine conditional expectations on returns.