More on matrices

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Recap:

$$AB \neq BA$$

$$m \times \underline{n} \quad \underline{n} \times p \qquad n \times p \quad m \times n$$

If

$$AB = BA$$

then A and B commute.

$$AI = IA = A$$

Inverse matrix

Definition

If A is a square matrix, a matrix B is called an *inverse* of A if

$n \times n$ $\quad$ $n \times n$

$$AB = I \text{ and } BA = I.$$

A matrix that has an inverse is called **invertible**.

$2 \cdot \frac{1}{2} = 1$

$\frac{1}{2} \cdot 2 = 1$

$$A = O = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$O \cdot B = O$

$B \cdot O = O$

the inverse of A

Uniqueness theorem

Theorem

If B and C are both inverses of A, then $B = C$.

"The inverse of A is unique"

The inverse of A exists. A^{-1}

$2, \frac{1}{2} = 2^{-1}$

Proof: Hypotheses:

$$\boxed{\begin{array}{ll} AB = I & BA = I \\ AC = I & CA = I \end{array}} \ (*)$$

We want to prove $B = C$

$$B = I \cdot B = \begin{cases} \overset{(*)}{=} A(AC)B \\ \overset{(*)}{=} (CA)B = C(AB) = C \cdot I = C \end{cases}$$

$\square$

If A is invertible then we can write

$$AA^{-1} = I, \quad A^{-1}A = I.$$

Is A^{-1} invertible? $(A^{-1})^{-1} = ?$

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}\,.}$$

Proof

Hypotheses

$$AA^{-1} = A^{-1}A = I$$
$$BB^{-1} = B^{-1}B = I$$

Goal

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(B^{-1}A^{-1}) \overset{\text{associative}}{=} A(BB^{-1})A^{-1} \overset{B \text{ is invertible}}{=} (A \cdot I)A^{-1} = AA^{-1} \overset{A \text{ is invertible}}{=} I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B \overset{B \text{ is invertible}}{=} I$$

$\square$

Transpose of a matrix

Definition

The *transpose* of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $B = (b_{ij})$ given by

$$b_{ij} = a_{ji}$$

The transpose of A is denoted by A^T.

$A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ $\quad B = (b_{ij}) \quad b_{ij} = a_{ji}$

B - $n \times m$ matrix $\quad B = (a_{ji})_{\substack{j=1,\dots,n \\ i=1,\dots,m}}$

$B = A^T$

$$\underset{2\times 3}{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \underset{3\times 2}{A^T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$\underset{2\times 2}{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

Theorem

Assume that α is a scalar and that A, B, and $\cancel{C}$ are matrices so that the indicated operations can be performed. Then:

(a) $(A^T)^T = A$;

(b) $(\alpha A)^T = \alpha(A^T)$;

(c) $(A+B)^T = A^T + B^T$;

(d) $(AB)^T = B^T A^T$.

Remember

$A = (a_{ij})$

$A^T = (a_{ji})$

$A^T = (\tilde{a}_{ij})$ $\quad \tilde{a}_{ij} = a_{ji}$

Proof

(a) $(A^T)^T = (\tilde{a}_{ij})^T = (\tilde{a}_{ji}) = (a_{ij}) = A$

(b) $(\alpha A)^T = (\alpha a_{ij})^T = (\alpha a_{ji}) = \alpha (a_{ji}) = \alpha A^T$ $\qquad \alpha A = (\alpha a_{ij})$

(c) $(A+B)^T = (a_{ij} + b_{ij})^T = (a_{ji} + b_{ji})$

$= (a_{ji}) + (b_{ji})$

$= A^T + B^T$

$A = (a_{ij})$

$B = (b_{ij})$

$A+B = (a_{ij} + b_{ij})$

Proof

$$(AB)^T$$

$$\underset{m\times n}{A} = (a_{ij})$$

$$\underset{n\times p}{B} = (b_{ij})$$

i,j - entry of $(AB)^T$

$= j,i$ - entry of (AB)

$$= \boxed{\sum_{k=1}^{n} a_{jk} b_{ki}}$$

ij - entry of AB

$$\sum_{k=1}^{n} a_{ik} b_{kj}$$

$$B^T = (\tilde{b}_{ij}) \quad p\times n$$

$$A^T = (\tilde{a}_{ij}) \quad n\times m$$

i,j - entry of $B^T A^T$

$$= \sum_{k=1}^{n} \tilde{b}_{ik} \tilde{a}_{kj}$$

$$= \sum_{k=1}^{n} b_{ki} a_{jk} = \boxed{\sum_{k=1}^{n} a_{jk} b_{ki}}$$

$$\Longrightarrow (AB)^T = B^T A^T$$

$\square$

Examples

Let A be invertible.

Then A^T is invertible and

$$\boxed{(A^T)^{-1} = (A^{-1})^T}$$

A is a square matrix

$$AA^{-1} = I$$
$$A^{-1}A = I$$

Remember

$$(AB)^T = B^T A^T \quad (*)$$

<u>Proof</u>

$$A^T \cdot (A^{-1})^T \stackrel{?}{=} I$$

$$(A^{-1})^T A^T \stackrel{?}{=} I$$

$$A^T \cdot (A^{-1})^T \stackrel{(*)}{=} (A^{-1}A)^T \underset{\text{invertible}}{\overset{A \text{ is}}{=}} I^T = I$$

$$(A^{-1})^T \cdot A^T \stackrel{(*)}{=} (AA^{-1})^T = I^T = I \qquad \square$$

A $m \times n$

A^T $n \times m$

A $n \times n$

A^T $n \times n$

Special type of square matrices

Definition

A matrix is said to be **symmetric** if $A^T = A$.

A must be square

- A is a $n \times n$ - matrix
- $A = (a_{ij})$ $\quad A^T = (a_{ji})$
- $A = A^T \iff \boxed{a_{ij} = a_{ji}}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$i = 1 \quad j = 1 \qquad a_{11} = a_{11}$

$i = 1 \quad j = 2 \qquad \boxed{a_{12} = a_{21}}$

$i = 2 \quad j = 1 \qquad \boxed{a_{21} = a_{12}}$

$i = 2 \quad j = 2 \qquad a_{22} = a_{22}$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

No!

$c_{23} \neq c_{32}$

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 7 & -3 & 2 \end{pmatrix}$$

Definition

A square matrix $A = (a_{ij})$ is said to be

upper triangular	if $a_{ij} = 0$ for $i > j$;
strictly upper triangular	if $a_{ij} = 0$ for $i \geq j$;
lower triangular	if $a_{ij} = 0$ for $i < j$;
strictly lower triangular	if $a_{ij} = 0$ for $i \leq j$;
diagonal	if $a_{ij} = 0$ for $i \neq j$.

$$A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & \ddots & \\ a_{11} & a_{12} & \cdots & a_{nn} \end{pmatrix}$$

upper triangular

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots \, 0 & a_{nn} \end{pmatrix}$$

strictly upper triangular

$$A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \end{pmatrix}$$

lower triangular

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & & 0 \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

strictly lower triangular

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & \ddots & 0 \\ a_{n1} & \cdots & 0 \end{pmatrix}$$

diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{nn} \end{pmatrix}$$

I_n diagonal matrix

upper triangular

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

strictly lower triangular

Theorem

The sum and product of two upper triangular matrices of the same size is upper triangular.

(1) Let A and B be upper triangular matrices.
Then, $A+B$ is upper triangular.

Hypotheses:
$a_{ij} = 0 \quad i > j$
$b_{ij} = 0 \quad i > j$

$$A+B = (a_{ij} + b_{ij})$$

If $i > j$, $a_{ij} + b_{ij} = 0 + 0 = 0$ $\square$

A is upper triangular

$a_{ij} = 0 \quad i > j$

(2) Let A and B be upper triangular matrices.
Then, AB is upper triangular.

Hypotheses:
$a_{ij} = 0 \quad i > j$
$b_{ij} = 0 \quad i > j$

i,j-entry of AB : $\sum_{k=1}^{n} a_{ik} b_{kj}$

$i > j$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \cdots a_{in} b_{nj} = 0$$

Note that each of these products have $a_{ik} = 0$ or $b_{kj} = 0$