RELATIVITY – MTH6132

PROBLEM SET 8

1. The metric for a particular 2-dimensional spacetime is given by

$$ds^2 = -e^{2Ar}dt^2 + dr^2$$

where A is an arbitrary constant. Calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric using the Euler-Lagrange equations.

2. Find the geodesic equations for cylindrical coordinates, $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$. Without explicitly solving the differential equations, explain why you already know what the solutions should be. **Hint**: There are only three nonzero Christoffel symbols.

3. In Lecture, we introduced Γ^a_{bc}, known as *Christoffel symbols of the second kind.* In this problem we will study *Christoffel symbols of the first kind*, defined by

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}\right).$$

- Write down the relationship between the two Christoffel symbols by raising (or lowering) the appropriate index.
- Show that the Christoffel symbols of the first kind are symmetric in the last two indices, i.e., $\Gamma_{abc} = \Gamma_{acb}$
- Show that if the metric g is diagonal, then

$$\Gamma^a_{ab} = \frac{\partial}{\partial x^b}\left(\frac{1}{2}\ln g_{aa}\right).$$

- The Kahn-Penrose metric, used to study colliding gravitational waves in local coordinates (u, v, x, y) has line element

$$ds^2 = 2du\,dv - (1-u)^2\,dx^2 - (1+u)^2\,dy^2,$$

  where it is assumed that $u \geq 0$ and $v < 0$. Find the Christoffel symbols for this metric.

4. Consider the two-dimensional space given in local coordinates $(x^1, x^2) = (x, y)$ by

$$ds^2 = e^y dx^2 + e^x dy^2.$$

(a) Write down the Lagrangian L for this metric.

(b) Employ the Euler-Lagrange equations to determine the following Christoffel symbols: $\Gamma^1{}_{11}$, $\Gamma^1{}_{12}$ and $\Gamma^2{}_{11}$ and $\Gamma^1{}_{22}$.

(c) Calculate the R^1_{212} component of the Riemann tensor. Compute also R_{1212}.

5. The Riemann curvature tensor of a certain manifold is of the form

$$R_{abcd} = K\left(g_{ac}g_{bd} - g_{ad}g_{cb}\right),$$

with K a constant. Show that:

(a) The tensor R_{abcd} defined above has the symmetries of the Riemann tensor discussed in the lecture;

(b) Observing that $\nabla_a g_{bc} = 0$, show that $\nabla_e R_{abcd} = 0$. What sort of surface could have a constant curvature?

(c) Show that the corresponding Ricci tensor is proportional to the metric, and that the Ricci scalar is a constant.

6. Suppose that the curvature of a spacetime satisfies the equation

$$R_{ab} - \frac{1}{2}R g_{ab} + \lambda g_{ab} = 0,$$

where λ is a constant. Show that the Ricci scalar satisfies $R = 4\lambda$. Using this result show then that $R_{ab} = \lambda g_{ab}$.

7. Show that the Christoffel connection satisfies

$$\Gamma^a{}_{ab} = \frac{1}{\sqrt{|g|}}\,\partial_b\sqrt{|g|}$$

where $|g| = |\det g_{ab}|$.

8. Consider the metric

$$ds^2 = -dt^2 + \frac{dr^2}{1 - k\,r^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

where k is a constant.

(a) Calculate the Christoffel symbols.

(b) Calculate the Ricci tensor and the Ricci scalar. **Hint:** use Problem 7 to simplify the calculation.

(c) Assuming that the Einstein tensor satisfies the equation:

$$G_{ab} = 8\,\pi\,T_{ab}\,,$$

find T_{ab}.

9. Considering two arbitrary vectors X^a and Y^a, apply the Leibniz rule to $T_{ab}X^aY^b$ to show that

$$[\nabla_c\,,\nabla_d\,]T_{ab} = -R^e{}_{acd}\,T_{eb} - R^e{}_{bcd}\,T_{ae}$$

10. Consider two arbitrary vectors X^a and Y^a. Given another vector Z^a, show that

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]} Z^a = R^a{}_{bcd} Z^b X^c Y^d \,,$$

where $\nabla_X = X^a \nabla_a$, $\nabla_Y = Y^a \nabla_a$ and $\nabla_{[X,Y]} = [X, Y]^a \nabla_a$.

11. Consider the general static spherically symmetric spacetime in four dimensions:

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,.$$

(a) Compute the Christoffel symbols.

(b) The components of the Riemann tensor.

(c) The components of the Ricci tensor.

12. Consider the following spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2 \, dv \, dr + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,,$$

where $M > 0$ is a constant.

(a) Compute the Christoffel symbols.

(b) The components of the Riemann tensor.

(c) The components of the Ricci tensor.