MTH5131 Actuarial Statistics

Coursework 3

This coursework is not to be turned in. You may ask questions about the coursework in tutorial or by email.

Exercise 1. The number of claims in a week arising from a certain group of insurance policies has a Poisson distribution with mean μ. Seven claims were incurred in the last week. The prior distribution of μ is uniform on the integers 8, 10 and 12.

1. Determine the posterior distribution of μ.

2. Calculate the Bayesian estimate of μ under squared error loss.

Exercise 2. For the estimation of a population proportion p, a sample of n is taken and yields x successes. A suitable prior distribution for p is Beta with parameters 4 and 4.

1. Show that the posterior distribution of p given x is Beta and specify its parameters.

2. Given that 11 successes are observed in a sample of size 25, calculate the Bayesian estimate under all-or-nothing (0/1) loss.

Exercise 3. A statistician wishes to find a Bayesian estimate of the mean of an exponential distribution with density function

$$f(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}$$

She is proposing to use a prior distribution of the form:

$$f(\mu) = \frac{\theta^\alpha e^{-\frac{\theta}{\mu}}}{\mu^{\alpha+1}\Gamma(\alpha)}, \mu > 0$$

You are given that the mean of this distribution is $\frac{\theta}{\alpha-1}$.

1. Write down the likelihood function for μ, based on a random sample of values $x_1, x_2, \ldots, x_n$ from an exponential distribution.

2. Find the form of the posterior distribution for μ and hence show that an expression for the Bayesian estimate for μ under squared error loss is:

$$\hat{\mu} = \frac{\theta + \sum x_i}{n + \alpha - 1}$$

3. Show that the Bayesian estimate for μ can be written in the form of a credibility estimate and write down a formula for the credibility factor.

4. The statistician now decides that she will use a prior distribution of this form with parameters $\theta = 40, \alpha = 1.5$.
   Her sample data have statistics $n = 100, \sum x_i = 9,826, \sum x_i^2 = 1,200,000$.
   Find the posterior estimate for μ and the value of the credibility factor in this case.

5. Comment on your results from part 4.

Exercise 4. Suppose that one observation is made on a $\mathsf{N}(\theta, 1)$ random variable and the observation equals 1. The prior on θ isn uninformative. What is the posterior probability that $\theta > 0$? Does this seem reasonable?

Exercise 5. Suppose the number of claims arising from a risk each year has a Poisson(λ) distribution. The prior distribution for λ is Gamma(α, β). Let μ be the prior mean.
You have been given data as shown in the first two columns in the table below.

Year	Number of Claims	Z at the start of year	$\overline{X}$ =average number of claims based on number of years of past data available at the start of year	At the start of the year, the credibility estimate of the number of claims in the coming year $= Z\overline{X} + (1-Z)\mu$
1	144			
2	144			
3	174			
4	148			
5	151			
6	156			
7	168			
8	147			
9	140			
10	161			

1. Assuming $\alpha = 100$, $\beta = 1$:
   (a) Using data in the first two columns, calculate the remaining columns.
   (b) Sketch the graph of Z, the credibility factor at the start of year (on the y-axis) against year (on the x-axis).
   (c) What does the graph imply about the reliability of the data in estimating the true but unknown expected number of claims for the risk?
   (d) On the same graph, plot the actual and credibility estimate of the number of claims (on the y-axis) against year (on the x-axis).
   (e) Comment on your results.
2. Assuming $\alpha = 500$, $\beta = 5$:
   (a) Using data in the first two columns, calculate the remaining columns.
   (b) Sketch the graph of Z, the credibility factor at the start of year (on the y-axis) against year (on the x-axis).
   (c) On the same graph, plot the actual and credibility estimate of the number of claims (on the y-axis) against year (on the x-axis).
   (d) Comment on your results, comparing against the results from 1.

Exercise 6. Show that in the Normal/Normal model in which the conditional distribution of X is N(θ, σ_1^2) and te prior distribution of θ is N(μ, σ_2^2), where μ, σ_1^2 and σ_2^2 are known, the posterior given the data $\underline{x}$ is

$$N\left(\frac{\mu\sigma_1^2 + n\sigma_2^2\bar{x}}{\sigma_1^2 + n\sigma_2^2}, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + n\sigma_2^2}\right)$$

where:

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}.$$