Theorem

Let A, B and C be matrices of the $\underbrace{\text{same size}}_{m\times n}$, and let α and β be scalars. Then: $\alpha, \beta \in \mathbb{R}$

(a) $A + B = B + A$; commutative

(b) $A + (B + C) = (A + B) + C$; associative

→ (c) $A + O = A$;

(d) $A + (-A) = O$, where $-A = (-1)A$;

→ (e) $\alpha(A + B) = \alpha A + \alpha B$;

(f) $(\alpha + \beta)A = \alpha A + \beta A$;

(g) $(\alpha\beta)A = \alpha(\beta A)$;

(h) $1A = A$.

$A \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} m$

n

a_{ij} row i, col. j

$V = \{A : A \text{ is a matrix of size } m \times n\} = \mathcal{M}_{m\times n}$

$\Longrightarrow$ $\boxed{\mathcal{M}_{m\times n} \text{ is a vector space}}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & & b_{mn} \end{pmatrix} \quad A+B = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

By the commutative property of real numbers, $= \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} & \cdots & b_{1n}+a_{1n} \\ \vdots & & & \vdots \\ b_{m1}+a_{m1} & \cdots & & b_{mn}+a_{mn} \end{pmatrix} = B+A$

$$A + (B + C) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ \vdots & \vdots & & \vdots \\ b_{m1}+c_{m1} & b_{m2}+c_{m2} & \cdots & b_{mn}+c_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & \cdots & a_{1n}+(b_{1n}+c_{1n}) \\ \vdots & \vdots & & \vdots \\ a_{m1}+(b_{m1}+c_{m1}) & a_{m2}+(b_{m2}+c_{m2}) & \cdots & a_{mn}+(b_{mn}+c_{mn}) \end{pmatrix}$$

Apply the associative property at the level of every entry

$$= \begin{pmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & \cdots & (a_{1n}+b_{1n})+c_{1n} \\ \vdots & \vdots & & \vdots \\ (a_{m1}+b_{m1})+c_{m1} & (a_{m2}+b_{m2})+c_{m2} & \cdots & (a_{mn}+b_{mn})+c_{mn} \end{pmatrix} = (A+B)+C$$

Proof

(c) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & b_{mn} \end{pmatrix}$ $\quad A + O = \begin{pmatrix} a_{11}+0 & a_{12}+0 & \cdots & a_{1n}+0 \\ \vdots & \vdots & & \vdots \\ a_{m1}+0 & a_{m2}+0 & & a_{mn}+0 \end{pmatrix} = A$

(d) $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ $\quad -A = (-a_{ij})$ $\quad A + (-A) = (a_{ij} - a_{ij}) = (0)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

(e) $\alpha(A+B) = \alpha A + \alpha B$ By the distributive property of real numbers

$\alpha(A+B) = \alpha(a_{ij} + b_{ij}) = (\alpha(a_{ij}+b_{ij}))$

$= (\alpha a_{ij} + \alpha b_{ij}) = (\alpha a_{ij}) + (\alpha b_{ij})$

$= \alpha A + \alpha B$

(f) $(\alpha+\beta)A = \alpha A + \beta A$

$((\alpha+\beta) a_{ij}) = (\alpha a_{ij} + \beta a_{ij}) = \alpha A + \beta A$

associative

(g) $(\alpha\beta)A = \alpha(\beta A)$ $\quad (\alpha\beta)A = ((\alpha\beta)a_{ij}) = (\alpha(\beta a_{ij})) = \alpha(\beta A)$

(h) $1A = A$

Examples and conclusion

$$2(A+3B) - 3(C+2B)$$

(e) + (g) distributive property

$$= 2A + 6B - 3C - 6B$$

(a) commutative $\overset{}{=}$ $2A + 6B - 6B - 3C$

(b) associative $=$ $2A + (6B - 6B) - 3C$

(d) $=$ $2A + O - 3C$

(c) 0-element of the sum $=$ $2A - 3C$

$$\underset{p\times q}{B} \quad \underset{m\times n}{A} \qquad q\,? \; m\,?$$

$$A X = B$$

?

A: matrix known

B: matrix known

$$\underset{m\times n}{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$a, b \in \mathbb{R}$

$ab = ba$

$AB \neq BA$ non commutative

$$\underset{m\times n}{A} \cdot \underset{p\times q}{B} \qquad \boxed{n=p}$$

Matrix multiplication

Definition

If $A = (a_{ij})$ is an $m \times \underline{n}$ matrix and $B = (b_{ij})$ is an $\underline{n} \times p$ matrix then the **product** AB of A and B is the $\underline{m \times p}$ matrix $C = (c_{ij})$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \,.$$

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{pmatrix}$$

$2 \times \underline{3}$ $\qquad$ $\underline{3} \times 4$

$$\underset{2\times 4}{AB} = \begin{pmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{pmatrix}$$

$3 \cdot 6 - 3 + 10$

BA

$3 \times \underline{4}$ $\quad$ $\underline{2} \times 3$

Definition

An **identity matrix** I is a square matrix with 1's on the diagonal and zeros elsewhere. If we want to emphasise its size we write I_n for the $n \times n$ identity matrix.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} n$$

n

$$a_{ii} = 1$$

$$a_{ij} = 0 \quad i \neq j$$

I_n

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$AI_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$I_2 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Theorem

Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $C = (c_{ij})_{n\times p}$ and $D = (d_{ij})_{n\times p}$ be matrices and $\alpha \in \mathbb{R}$. Then,

(a) $(A+B)C = AC + BC$ and $A(C+D) = AC + AD$;

(b) $\alpha(AC) = (\alpha A)C = A(\alpha C)$;

(c) $I_m A = A I_n = A$; A $m\times n$

Let $X = (x_{ij})_{m\times n}$, $Y = (y_{ij})_{n\times p}$ and $Z = (z_{ij})_{p\times q}$. Then,

(d) $(XY)Z = X(YZ)$.

Proof (a) Distributive property $M = A+B = (m_{ij})$

$\boxed{m_{ij} = a_{ij} + b_{ij}}$

$(A+B)C = MC$

~~ar~~ ij-entry of MC $\sum_{k=1}^{n} m_{ik} c_{kj} = \sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj}$

By the distributive property of real numbers $= \sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj}$

$=$ ij-entry of AC + ij-entry of BC

It follows that

$(A+B)C = AC + BC$

Proof

(b) $\alpha(AC) = (\alpha A)C = A(\alpha C)$

ij-entry of AC : $\sum_{k=1}^{n} a_{ik} c_{kj}$

ij-entry of $\alpha(AC)$: $\alpha \sum_{k=1}^{n} a_{ik} c_{kj} = \underbrace{\sum_{k=1}^{n} (\alpha a_{ik}) c_{kj}}_{\text{ij-entry of } (\alpha A)C} = \underbrace{\sum_{k=1}^{n} a_{ik} (\alpha c_{kj})}_{\text{ij-entry of } A(\alpha C)}$

(c) A $m \times n$ matrix

$$\underset{m\times m}{I_m} \underset{m\times n}{A} = \underset{m\times n}{A} \underset{n\times n}{I_n} = A$$

$I_m = (l_{ij})$ $\quad l_{ij} = 0 \ i \neq j$, $\quad l_{ij} = 1 \ i = j$

ij-entry of $I_m A$: $\sum_{k=1}^{m} l_{ik} a_{kj} = l_{ii} a_{ij} = a_{ij} \Longrightarrow I_m A = A$

ij-entry of $A I_n$: $\sum_{k=1}^{n} a_{ik} l_{kj} = a_{ij} l_{jj} = a_{ij} \Longrightarrow A I_n = A$

Proof

$X = (x_{ij})_{m \times n} \qquad Y = (y_{ij})_{n \times p} \qquad Z = (z_{ij})_{p \times q}$

$$(XY)Z = X(YZ)$$

Let $XY = T \qquad T = (t_{ij})_{m \times p} \qquad t_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$

$$= x_{i1} y_{1j} + x_{i2} y_{2j} + \cdots + x_{in} y_{nj}$$

TZ ij-entry of TZ $= \sum_{k=1}^{p} t_{ik} z_{kj} = \sum_{h=1}^{p} t_{ih} z_{hj}$

$$= \sum_{h=1}^{p} \underbrace{\left(\sum_{k=1}^{n} x_{ik} y_{kh} \right)}_{t_{ih}} z_{hj}$$

by using the distributive and associative property at the level of the entries $= \sum_{k=1}^{n} x_{ik} \underbrace{\sum_{h=1}^{p} y_{kh} z_{hj}}$ $=$ ij-entry of $X(YZ)$

$\square$