Vectors & Matrices

Solutions to Problem Sheet 6

1. Matrix A has size 2×3, and B has size 3×2. By the definition of matrix multiplication, the product AB will be a matrix of size 2×2. To compute the elements of this matrix, we use the formula given in the notes,

$$
\begin{aligned}
AB &= \begin{pmatrix} 5 & -2 & 1 \\ -1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 8 \\ -4 & 2 \\ -7 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 5 \cdot 1 + (-2) \cdot (-4) + 1 \cdot (-7) & 5 \cdot 8 + (-2) \cdot 2 + 1 \cdot 2 \\ (-1) \cdot 1 + 3 \cdot (-4) + (-3) \cdot (-7) & (-1) \cdot 8 + 3 \cdot 2 + (-3) \cdot 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 38 \\ 8 & -8 \end{pmatrix} .
\end{aligned}
$$

Similarly, the product AC will also give a 2×2 matrix, with entries computed by

$$
\begin{aligned}
AC &= \begin{pmatrix} 5 & -2 & 1 \\ -1 & 3 & -3 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 7 & 1 \\ 1 & -5 \end{pmatrix} \\
&= \begin{pmatrix} 5 \cdot (-3) + (-2) \cdot 7 + 1 \cdot 1 & 5 \cdot (-2) + (-2) \cdot 1 + 1 \cdot (-5) \\ (-1) \cdot (-3) + 3 \cdot 7 + (-3) \cdot 1 & (-1) \cdot (-2) + 3 \cdot 1 + (-3) \cdot (-5) \end{pmatrix} \\
&= \begin{pmatrix} -28 & -17 \\ 21 & 20 \end{pmatrix} .
\end{aligned}
$$

In order for the sum $B + C$ to be defined, we require matrices B and C to have matching size. In fact, both B and C have dimensions 3×2, and so this sum exists. Using the formulation given in the lecture notes, the elements of the resulting matrix are given by

$$
B + C = \begin{pmatrix} 1 & 8 \\ -4 & 2 \\ -7 & 2 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 7 & 1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 + (-3) & 8 + (-2) \\ -4 + 7 & 2 + 1 \\ -7 + 1 & 2 + (-5) \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 3 & 3 \\ -6 & -3 \end{pmatrix} .
$$

To compute the matrix $A(B + C)$, we can use the distributivity property of matrix multiplication,

$$A(B+C) = AB + AC \ ,$$

shown in Theorem 7.1.14. We have already evaluated matrices AB and AC, and so

$$A(B+C) = AB+AC = \begin{pmatrix} 6 & 38 \\ 8 & -8 \end{pmatrix} + \begin{pmatrix} -28 & -17 \\ 21 & 20 \end{pmatrix} = \begin{pmatrix} 6+(-28) & 38+(-17) \\ 8+21 & (-8)+20 \end{pmatrix} = \begin{pmatrix} -22 & 21 \\ 29 & 12 \end{pmatrix} .$$

A similar property can be used to evaluate $2A(C+B)$. Firstly, note that by the property of the commutativity of matrix addition (seen in Theorem 7.1.8), we have $C+B = B+C$, and so $2A(C+B) = 2A(B+C)$. We now note that we have computed the matrix $A(B+C)$ above, and so the only operation that we need to use is scalar multiplication. We obtain

$$2A(C+B) = 2A(B+C) = 2\begin{pmatrix} -22 & 21 \\ 29 & 12 \end{pmatrix} = \begin{pmatrix} -44 & 42 \\ 58 & 24 \end{pmatrix} .$$

2. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ (in other words, the element of A at the i^{th} row and j^{th} column is equal to a_{ij}, and the corresponding element of B is b_{ij}). We begin by showing that the matrices on both sides of the equation have matching sizes. As A and B both have size $m \times n$, so does the sum $A+B$. Scalar multiplication does not change the dimensions of a matrix, and so $\alpha(A+B)$ also has size $m \times n$.

   The fact that scalar multiplication does not change the dimensions of a matrix also tells us that αA and αB both have size $m \times n$, and their sum, $\alpha A + \alpha B$ therefore also has size $m \times n$. This shows that the matrices on both sides of the equation have matching size. All that remains is to show that they have matching elements.

   By the definition of matrix addition, the elements of matrix $A+B$ are given by

$$A+B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij}+b_{ij})_{m\times n} \ .$$

   By the definition of the scalar multiple of a matrix, the elements of $\alpha(A+B)$ are given by

$$\alpha(A+B) = \alpha(a_{ij}+b_{ij})_{m\times n} = (\alpha(a_{ij}+b_{ij}))_{m\times n} \ .$$

Given that, for each i and j, the values a_{ij} and b_{ij} are real numbers, we can use the property of distributivity over the real numbers to see that

$$\alpha(A+B) = (\alpha(a_{ij}+b_{ij}))_{m\times n} = (\alpha\cdot a_{ij}+\alpha\cdot b_{ij}))_{m\times n} \ .$$

We can once again invoke the definition of scalar multiplication over matrices to get

$$\begin{aligned} \alpha A &= \alpha(a_{ij})_{m\times n} = (\alpha\cdot a_{ij})_{m\times n} \ , \\ \alpha B &= \alpha(b_{ij})_{m\times n} = (\alpha\cdot b_{ij})_{m\times n} \ . \end{aligned}$$

Using these formulations, we compute the sum $\alpha A + \alpha B$ as

$$\alpha A + \alpha B = (\alpha\cdot a_{ij})_{m\times n} + (\alpha\cdot b_{ij})_{m\times n} = (\alpha\cdot a_{ij}+\alpha\cdot b_{ij})_{m\times n} \ .$$

Matrices $\alpha(A+B)$ and $\alpha A+\alpha B$ have the same sizes and same elements, and are therefore (by the definition of matrix equality) equal.

3. To evaluate A^{2024}, we compute small powers of A and aim to identify a pattern.

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0+(-1)\cdot 1 & 0\cdot(-1)+(-1)\cdot 0 \\ 1\cdot 0+0\cdot 1 & 1\cdot(-1)+0\cdot 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)I_2 \ .$$

The effect of multiplying a 2×2 matrix by I_2 is for the matrix to remain the same. Since A^2 has been shown to equal $(-1)I_2$, the effect of multiplying a 2×2 matrix by A^2 is for each element to be multiplied by the scalar factor of -1. Hence,

$$\begin{aligned} A^3 &= A^2A \\ &= ((-1)I_2)A && \text{(from above)} \\ &= (-1)(I_2A) && \text{(by Theorem 7.1.14 b))} \\ &= (-1)A && \text{(by Theorem 7.1.14 c))} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \end{aligned}$$

Similarly, we can use the fact that $A^2 = (-1)I_2$ to evaluate the matrix A^4,

$$\begin{aligned}
A^4 &= A^2A^2 \\
&= ((-1)I_2)((-1)I_2) && \text{(from above)} \\
&= (-1)(-1)(I_2I_2) && \text{(by Theorem 7.1.14 b))} \\
&= I_2 && \text{(by Theorem 7.1.14 c))} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .
\end{aligned}$$

This tells us that $A^4 = A^0$, and that all further powers of A will cycle through the values A, $-I_2$, $-A$ and I_2. Indeed,

$$A^5 = A^4A = I_2A = A \ .$$

Therefore, we can evaluate any power A^n (for any $n \in \mathbb{N}$) by identifying the remainder of n after division by 4. If 4 divides n, then $A^n = I_2$. If n has remainder of 1 after division by 4, then $A^n = A$. If the remainder is 2, then $A^n = -I_2$, and a remainder of 3 gives $A^n = -A$. Since $2024 = 4 \times 506$, 4 divides 2024, and so $A^{2024} = I_2$.

4. Suppose, for the sake of contradiction, that there is a second (and distinct) matrix J_n with the property that

$$J_nA = AJ_n = A$$

for any $n \times n$ matrix A. If we let this A be the (original) $n \times n$ identity matrix I_n, we would have

$$J_nI_n = I_nJ_n = I_n \ .$$

However, since I_n is also an identity matrix, then

$$J_nI_n = I_nJ_n = J_n \ ,$$

and so, combining these equalities, we find that $I_n = J_n$. Since we assumed that I_n and J_n were distinct, and so could not be equal, we have derived a contradiction. Hence, identity matrices for $n \times n$ systems are unique.

5. (i) By the definition of matrix multiplication, we have

$$AB = \begin{pmatrix} 2 & 1 \\ 6 & z \end{pmatrix} \begin{pmatrix} z & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} 2z + 1 \cdot (-6) & 2 \cdot (-1) + 1 \cdot 2 \\ 6z - 6z & 6 \cdot (-1) + 2z \end{pmatrix} = \begin{pmatrix} 2z - 6 & 0 \\ 0 & 2z - 6 \end{pmatrix} .$$

Similarly,

$$BA = \begin{pmatrix} z & -1 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & z \end{pmatrix} = \begin{pmatrix} 2z + (-1) \cdot 6 & z - z \\ (-6) \cdot 2 + 2 \cdot 6 & (-6) \cdot 1 + 2z \end{pmatrix} = \begin{pmatrix} 2z - 6 & 0 \\ 0 & 2z - 6 \end{pmatrix} .$$

Hence, $AB = BA = (2z - 6)I_2$.

(ii) By definition, for a matrix C to be an inverse of A we need $AC = CA = I_2$. We have seen in part (i) that the matrix B has the property that $AB = BA = (2z - 6)I_2$. If we define

$$C = \left(\frac{1}{2z - 6}\right) B = \frac{1}{2z - 6} \begin{pmatrix} z & -1 \\ -6 & 2 \end{pmatrix} ,$$

then we have

$$\begin{aligned} AC &= A\left(\left(\frac{1}{2z-6}\right) B\right) \\ &= \left(\frac{1}{2z-6}\right)(AB) && \text{(by Theorem 7.1.14 b))} \\ &= \left(\frac{1}{2z-6}\right)(2z-6)I_2 && \text{(from above)} \\ &= I_2 , \end{aligned}$$

and

$$\begin{aligned} CA &= \left(\left(\frac{1}{2z-6}\right) B\right) A \\ &= \left(\frac{1}{2z-6}\right)(BA) && \text{(by Theorem 7.1.14 b))} \\ &= \left(\frac{1}{2z-6}\right)(2z-6)I_2 && \text{(from above)} \\ &= I_2 . \end{aligned}$$

Therefore, C is the inverse of A.

(iii) From part (ii), the inverse of A, if it exists, is given by

$$C = \frac{1}{2z-6}\begin{pmatrix} z & -1 \\ -6 & 2 \end{pmatrix} .$$

However, if $2z-6$ were equal to zero, then this expression would be undetermined, and so A would have no inverse. This happens when $z=3$, for which

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} .$$

Notice that the second row of this matrix is equal to the first row multiplied by a factor of 3. This gives us a clue as to why the system would not be invertible. Indeed, if we apply this matrix to a vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} ,$$

we would get

$$A\mathbf{x} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 6x+3y \end{pmatrix} .$$

It is clear that the resulting vector must be of the form

$$A\mathbf{x} = \begin{pmatrix} k \\ 3k \end{pmatrix} ,$$

for some $k \in \mathbb{R}$. Therefore, any vector not in this form could not exist in the image of the matrix A, and thus, A is not invertible.

6. Suppose we define a matrix $B = (b_{ij})_{n\times n}$. We see that

$$AB = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^n b_{k1} & \sum_{k=0}^n b_{k2} & \dots & \sum_{k=0}^n b_{kn} \\ \sum_{k=0}^n b_{k1} & \sum_{k=0}^n b_{k2} & \dots & \sum_{k=0}^n b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^n b_{k1} & \sum_{k=0}^n b_{k2} & \dots & \sum_{k=0}^n b_{kn} \end{pmatrix} .$$

Hence, the product AB is given by

$$AB = \left(\sum_{k=0}^{n} b_{kj} \right)_{n \times n} .$$

Note that this formulation does not vary according to the index i, and hence, does not change as we go down the rows of matrix AB. Therefore, all rows of the matrix AB are identical. In order for B to be the inverse of A, we would need $AB = I_n$, and so

$$\sum_{k=0}^{n} b_{k1} = 1 ,$$

since the $(1, 1)$ entry of $AB = I_n$ should be 1. However, moving down to the second row, we see that the $(2, 1)$ entry of $AB = I_n$ should be 0, but this would give

$$\sum_{k=0}^{n} b_{k1} = 0 ,$$

contradicting the equality above. Hence, there is no B such that $AB = I_n$, and so A is not invertible.