Chapter 1-2. Basics of logic in math

Chapter 3-5 Vectors.

- Vector geometric interpretation
- objects in geometry. Eg. lines points.
- Cartesian equations of geometry objects
- Scalar product
- Vector product
- Equations plane, line. Distance between objects; intersections.

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Chapter 6 linear systems (coefficient/Augmented matrix; Gassian/Gauss Jordan Algoritm)

Chapter 7 matrix.

last lecture. 7.1.1 — 7.1.5

Example 7.6

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \\ 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -2 & 1 \end{pmatrix}$$

$$2A+3B = \begin{pmatrix} 4 & 9 \\ -1\cdot 2+3\cdot 2 & 2\cdot 2+3\cdot 3 \\ 4\cdot 2-2\cdot 3 & 0+3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 4 & 13 \\ 2 & 3 \end{pmatrix}$$

Definition 7.1.7 <u>zero matrix</u> $O_{m\times n}$ or simple O

for the $m\times n$ matrix, all of whose entries are zero.

Theorem 7.1.8. Let A, B, and C be matrices of the same size $m\times n$, let α and β be scalars.

Then.

a) $A+B = B+A$	addition is commutative
b) $A+(B+C) = (A+B)+C$	addition is associative
c) $A+O = A$	zero matrix is the identity for the addition
d) $A+(-A) = O$, where $-A = (-1)\cdot A$	$-A$ is the additive inverse of A
e) $\alpha(A+B) = \alpha A + \alpha B$	The multiplication of matrices by scalar is distributive
f) $(\alpha+\beta)\cdot A = \alpha A + \beta\cdot A$	distributive property
g) $(\alpha\cdot\beta)\cdot A = \alpha(\beta A)$	The multiplication by scalars is associative
h) $1A = A$	1 is the identity for the multiplication by scalar.

Z

Proof. b) $A+(B+C)=(A+B)+C$

LHS. $B+C$ is $m \times n$ matrix. $A+(B+C)$ is also $m \times n$ matrix. The ij entry of $B+C$ is $b_{ij}+c_{ij}$, so the ij entry of $A+(B+C)$ $a_{ij}+b_{ij}+c_{ij}$.

RHS. Similary, $A+B$ is $m \times n$ matrix. $(A+B)+C$ is a $m \times n$ matrix. The ij entry of $A+B$ is $a_{ij}+b_{ij}$. and The ij entry of $(A+B)+C$ is $a_{ij}+b_{ij}+c_{ij}$

Equality. Thus. $A+(B+C)$ has the same size as $(A+B)+C$. with ~~et~~ the same entries for the two matrices. (Definition 2.1.5 Equality)

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Example 7.1.9 Simplify $2(A+3B)-3(C+2B)$, where $A, B, C,$ are matrices with the same size.

Solution

$$\begin{aligned}2(A+3B)-3(C+2B) &= 2A+6B-3C-6B\\ &= 2A-3C+6B-6B\\ &= 2A-3C+6(B-B)\\ &= 2A-3C+6\,O\\ &= 2A-3C+O\\ &= 2A-3C\end{aligned}$$

Remark 7.1.10. From Theorem 7.1.8. We have the set $A_{m\times n}$ of all matrices of size $m\times n$. together with the operation of <u>addition of matrices</u> and <u>multiplication by real scalar</u> is a vector space

(Check the Definition 3.2.2)

Definition 7.1.11 (Matrix multiplication) If $A = (a_{ij})$ is $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix then the product AB of A and B is the $m \times p$ matrix $C = (c_{ij})$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Example 7.1.12 Compute $(1,3)$ entry and $(2,4)$ entry of AB. Where

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{pmatrix}$$

Sol:

$(1,3)$ entry: $\quad 3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = 25$

$(2,4)$ entry $\quad 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$

AB is a matrix with 2×4

Definition 7.1.13 (Identity matrix)

An identity matrix I is a square matrix with 1's on the diagonal and 0's else where. If we want to emphasie its size, we write I_n for $n \times n$ identity matrix.

$$I_n = \begin{array}{c} \\ 1 \\ \vdots \\ \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & & \cdots & & n \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & & & \vdots \\ & & 1 & & \\ \vdots & & & 1 & \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Theorem 7.1.14. Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $C = (c_{ij})_{n\times p}$, $D = (d_{ij})_{n\times p}$ be matrice and $\alpha \in \mathbb{R}$. Then

a) $(A+B)\cdot C = AC + BC$ and

$$A(C+D) = A\cdot C + AD.$$

b) $\alpha(A\cdot C) = (\alpha A)C = A(\alpha C)$

c) $I_m A = A\cdot I_n = A$

d) Let $X = (x_{ij})_{m\times n}$ $Y = (y_{ij})_{n\times p}$ $Z = (z_{ij})_{p\times q}$

Then $(X\cdot Y)\cdot Z = X(Y\cdot Z)$

Proof. a). $(A+B)\cdot C = AC + B\cdot C$

Let $A+B = M = (m_{ij})_{m\times n}$ so $m_{ij} = a_{ij} + b_{ij}$.

Size LHS: Thus MC has the size as $m\times p$.

Size RHS: Aslo AC and BC are matrices with size $m\times p$

Thus $AC+BC$ is an $m\times p$ matrix

the ij entries of LHS and RHS equal: Then ij entrix of MC is

$$\sum_{k=1}^{n} m_{ik}\cdot c_{kj} = \sum_{k=1}^{n} (a_{ik} + b_{ik})\cdot c_{kj}$$

$$= \sum_{k=1}^{n} (a_{ik}\cdot c_{kj} + b_{ik}\cdot c_{kj})$$

$$= \boxed{\sum_{k=1}^{n} a_{ik}\cdot c_{kj}} + \boxed{\sum_{k=1}^{n} b_{ik}\cdot c_{kj}}$$

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$= A\cdot C + B\cdot C$

Thus we complete the ~~proof~~ of Theorem 7.1.14 a) by showing the size of LHS and RHS matrices equal and the i,j entries equal as well.

c). $I_m A = A \cdot I_n = A$

proof. $\underset{m\times m}{I_m} A_{m\times n}$ has the size $m\times n$.

$\underset{m\times n}{A} \cdot \underset{n\times n}{I_n}$ has the size $m\times n$

The i,j entry of $I_m \cdot A$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

I_{ii}

$$0\times a_{1j} + 0\times a_{2j} + \dots + 1\cdot a_{ij} + \dots + 0\times a_{mj} = a_{ij}$$

The ij entry of $A\cdot I_n$ is

I_{jj}

$$a_{i1}\times 0 + a_{i2}\times 0 + \dots + a_{ij}\times 1 + \dots + a_{in}\times 0$$

Thus. The i,j entries of $I_m A$, $A I_n$ and $A = a_{ij}$. equal.

d) $X = (x_{ij})_{m\times n}$, $Y = (y_{ij})_{n\times p}$, $Z = (z_{ij})_{p\times q}$

Then, $(X\cdot Y)Z = X(Y\cdot Z)$

proof. LHS $X\cdot Y = T = (t_{ij})_{m\times p}$ So,

$$t_{ij} = x_{i1}y_{1j} + x_{i2}\cdot y_{2j} + \dots + x_{in}\cdot y_{nj} = \sum_{k=1}^{n} x_{ik}\cdot y_{kj}$$

Now $(X\cdot Y)\cdot Z = \underset{m\times p}{T}\,\underset{p\times q}{Z}$ has the size $m\times q$

Thus the ij entry of $(XY)Z$ is

$$\sum_{k=1}^{p} t_{ik} z_{kj} = (x_{i1}\cdot y_{11} + x_{i2}\cdot y_{21} + \dots + x_{in} y_{n1})\cdot z_{1j} \quad k=1$$
$$+ (x_{i1} y_{12} + x_{i2} y_{22} + \dots + x_{in} y_{n2})\cdot z_{2j} \quad k=2$$
$$\vdots$$
$$+ (x_{i1} y_{1p} + x_{i2} y_{2p} + \dots + x_{in} y_{np})\cdot z_{pj} \quad k=p$$

RHS. $\underset{n\times p}{Y}\,\underset{p\times q}{Z}$ has the size $n\times q$

$\underset{m\times n}{X}\cdot \underset{n\times q}{(Y\cdot Z)}$ has the size $m\times q$

$$= \sum_{r=1}^{n}\sum_{s=1}^{p} x_{ir}\, y_{rs}\, z_{sj}$$

We can do the same to get the i,j entries $X\cdot(Y\cdot Z)$.

Thus. The LHS. RHS matrices have the same size. and the same i,j entries.

Homework: Why $AB \neq BA$ in general

When will $AB = BA$