last lecture 6.2

- Gaussian algorithm — any Matrix → row echelon format
- Gauss-Jordan algorith → reduced row echelon

6.3. special classes of linear system

Definition. A $m \times n$ linear syster

- Overdermined if $m > n$
- underdertmined if $m < n$

m: # equations. n: unknown variable

▷ Usnallly overdetermined systems are inconsistant which means no solution.

▷ Underdertermined system $\begin{cases} \text{no solution. inconstant} \\ \text{infinite many solutions.} \end{cases}$

Theorem 6.3.2. If an underdetermined system is consistent, it must have infinitely many solution

proof.

(m rows)

Note that we know any matrix can be transformed to the row echelon form. (r non zero rows)

thus we have $r \le m$. Thus there are r leading variables and consequently $n-r \geq n-m > 0$, free variables.

Definition 6.3.3 A linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad + \vdots = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases}$$

is said to homogeneous if $b_i = 0$ for all i. Otherwise it is said to be inhomogeneous.

$$\left(A \,\middle|\, \begin{matrix} b_i \\ \vdots \\ b_n \end{matrix} \right) \longrightarrow \left(A \,\middle|\, \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right)$$

Associated homogeneous system

Example 6.3.4.

$$\begin{cases} 3x_1 + 2x_2 + 3x_3 = 2 \\ 2x_1 - x_2 + x_3 = 5 \end{cases}$$

$\Rightarrow$ The associated homogeneous system

$$\begin{cases} 3x_1 + 2x_1 + 5x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$$

$m \times n$

$$\begin{cases} \sum a_{ij} x_j = 0 \\ \vdots \\ \sum a_{mj} x_j = 0 \end{cases}$$

Any homogeneous linear syste always have a solution. so called trivial or zero solution $(0, 0, 0, \ldots 0)$.

Theorem 6.3.5. An underdetermined homogeneous system always has non-trivial solutions.

Proof. We know a homogeneous system is consistent. Thus if the systen is underdetermined and homogeneou it must have infinitely many solution by theorem 6.3.2. Thus it must have a non-zero (non-trivial) solution.

Now we look at a special case of $n \times n$ system.

Theorem 6.3.6 An $n \times n$ system is consistent and has a unique solution. if and only if the only solution of the associated homogeneous system is the zero solution

Proof. Follows two obersvation.

- The same sequence of elementary row operation that brings the argumented matrix of a system to row echelon form (Gaussian Algorithm), also bring ~~the ag~~ ~~ag~~ augmented matrix of the associated homogenous system to row echelon form.
- A $n \times n$ system in row echelon form has a unique solution if there are n leading varable

Thus. if an $n \times n$ system is consitenten and has a unique soluton. the corresponding homogeneous system must also have an unique solution. which is necessarily the zero soluton.

Chapter 7

Matrices

7.1. Matrices and basic properties

Definition 7.1.7. An $m \times n$ matrix A is a rectangular array of scalar (real numbers)

$$\begin{pmatrix} a_{11} \cdots & a_{1n} \\ \vdots & \\ a_{m1} \cdots & a_{mn} \end{pmatrix}$$

We write $A = (a_{ij})_{m\times n}$ or $A = (a_{ij})$ to denote an $m \times n$ matrix whose (i,j)-entry is a_{ij}. if $m \times n$ is to be square, it means $m = n$., we have $n \times n$ matric

Definition 7.1.3 Two matrices A and B are equal. we write $A = B$ if they have the same size and $a_{ij} = b_{ij}$, where $A = (a_{ij})$ and $B = (b_{ij})$

Definition If $A = (a_{ij})_{m\times n}$ and α is a scalar. the αA is the $m \times n$ matrix whose (i,j)-entry is

αa_{ij}

Definition 7.1.5. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m \times n}$ the sum $A+B$ is the $m \times n$ matrice whose (i,j)-entry $a_{ij} + b_{ij}$

Example 7.1.6

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \\ 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -2 & 1 \end{pmatrix}$$

$$3A + 2B =$$