Mathematical Tools of Asset Management MTH6134

Capital Asset Princing Model (CAPM)

Dr. Melania Nica

Plan

- Last week
  - Mean Variance Portfolio Theory
  - Choosing an Optimal Efficient Portfolio without Risk Free Asset
- Today
  - Introduce Risk Free assets in our framework
  - CAPM
    - Security Market Line
    - Security Market Line versus Capital Market Line
    - Systematic Risks vs Unsystematic Risks
    - The pros and the cons of CAPM

Optimal Efficient Portfolio with risk free asset

Portfolio P consisting of a proportion w invested in a risky asset i and $(1 - w)$ in risk-free asset with sure return r

$$E(R_P) = wE(R_i) + (1 - w)r$$

$$w = \frac{E(R_P) - r}{E(R_i) - r}$$

Hence:

$$Var(R_P) = w^2 Var(R_i) = \left(\frac{E(R_P) - r}{E(R_i) - r}\right)^2 Var(R_i)$$

Optimal Efficient Portfolio with risk free asset

In terms of standard deviation:

$$\sigma_P = \frac{E(R_P) - r}{E(R_i) - r}\sigma_i$$

Expected return of portfolio P:

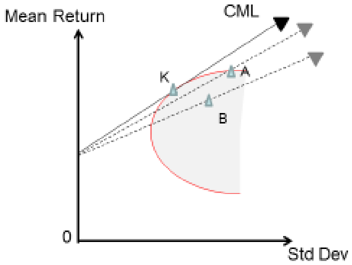
$$E(R_P) = r + [E(R_i) - r]\frac{\sigma_P}{\sigma_i}$$

$E(R_P)$ is a straight line in the return - standard deviation space

Optimal Efficient Portfolio with risk free asset

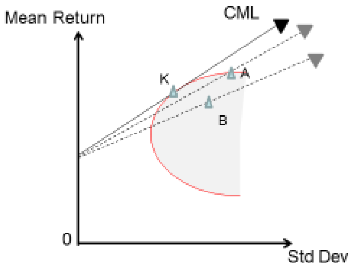
Capital Market Line for an asset *i* ($\mathbf{CML}_i$): the slope of a portfolio consisting of a risk free asset and risky asset *i*

$$E(R_P) = r + \frac{[E(R_i) - r]}{\sigma_i}\sigma_P$$



Tangency Portfolio (K):CML tangent to efficient MV frotier

Optimal Efficient Portfolio with risk free asset



Efficient frontier: CML$_K$ - all assets (securities/portfolios) formed with K and risky free asset.

$$E\left(R_P\right) = r + \left[E_K - r\right] \frac{\sigma_P}{\sigma_K}$$

E_K: expected return of tangency portfolio

Capital Asset Pricing Model (CAPM)

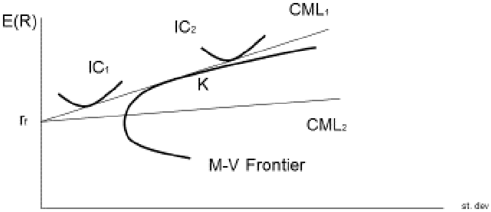
Capital Market Equilibrium: demand for risky securities = supply of risky securities

Assumptions

- Investors maximise their utility defined over expected return and return variance
- Unlimited amounts may be borrowed or loaned at risk free rate
- Investors might have homogeneous expectations regarding future assets returns
- Asset markets are perfect and frictionless: no taxes on sales or purchases, no transaction costs and no short sales restrictions

Capital Asset Pricing Model (CAPM)

Investors have different attitude towards risk



Investor 2 is less risk averse than investor 1

Two Fund Separation Theorem
Any risk averse investor can form their portfolio by combining two assets:

- the tangency portfolio of risky assets and the risk free rate
  - the degree of risk aversion dictates the portfolio weights placed on each fund

CAPM - A mathematical representation

Demand for securities:

Let's form portfolio P from arbitrary risky portfolio i and the tangency portfolio K with portfolio weights are w and respectively $1 - w$.

$$E(R_P) = wE(R_i) + (1 - w)E(R_K)$$

$$\sigma_P^2 = w^2\sigma_i^2 + (1 - w)^2\sigma_K^2 + 2w(1 - w)\sigma_{iK} \Leftrightarrow$$
$$\sigma_P = \left(w^2\sigma_i^2 + (1 - w)^2\sigma_K^2 + 2w(1 - w)\sigma_{iK}\right)^{1/2}$$

CAPM - A mathematical representation

The slope of capital market line at the tangency point, CML_K:

$$\frac{E(R_K) - r}{\sigma_K}$$

Slope of the MV frontier at K is:

$$\lim_{w \to 0} \frac{dE(R_P)}{d\sigma_P}$$

Hence, we need:

$$\lim_{w \to 0} \frac{dE(R_P)}{d\sigma_P} = \frac{E(R_K) - r}{\sigma_K}$$

CAPM - A mathematical representation

Now, let's focus on $\lim_{w \to 0} \frac{dE(R_P)}{d\sigma_P}$:

$$\frac{dE(R_P)}{d\sigma_P} = \frac{dE(R_P)/dw}{d\sigma_P/dw}$$

With:

$$\frac{dE(R_P)}{dw} = E(R_i) - E(R_K)$$

CAPM - A mathematical representation

Further,

$$\sigma_P = \left(w^2 \sigma_i^2 + (1-w)^2 \sigma_K^2 + 2w(1-w)\sigma_{iK} \right)^{1/2}$$

$$\begin{aligned} \lim_{w \to 0} \frac{d\sigma_P}{dw} &= \frac{1}{2} \left(\sigma_K^2\right)^{-1/2} \left(-2\sigma_K^2 + 2\sigma_{iK}\right) \\ &= \frac{\sigma_{iK} - \sigma_K^2}{\sigma_K} \end{aligned}$$

Thus

$$\lim_{w \to 0} \frac{d\sigma_P}{dw} = \frac{1}{2} \left(\sigma_K^2\right)^{-1/2} \left(-2\sigma_K^2 + 2\sigma_{iK}\right) = \frac{\sigma_{iK} - \sigma_K^2}{\sigma_K}$$

CAPM - A mathematical representation

Thus the slope of the MV frontier at K is:

$$\lim_{w \to 0} \frac{dE\left(R_P\right)}{d\sigma_P} = \frac{E\left(R_i\right) - E\left(R_K\right)}{\frac{\sigma_{iK} - \sigma_K^2}{\sigma_K}} = \frac{\sigma_K\left(E\left(R_i\right) - E\left(R_K\right)\right)}{\sigma_{iK} - \sigma_K^2}$$

CAPM - A mathematical representation

The slope of capital market line at K:

$$\frac{E(R_K) - r}{\sigma_K}$$

Thus the optimum portfolio is given by:

$$\frac{E(R_K) - r}{\sigma_K} = \frac{\sigma_K (E(R_i) - E(R_K))}{\sigma_{iK} - \sigma_K^2}$$

$$(E(R_K) - r)\left(\sigma_{iK} - \sigma_K^2\right) = \sigma_K^2 (E(R_i) - E(R_K))$$

$$\begin{aligned}
(E(R_K) - r)\left(\sigma_{iK} - \sigma_K^2\right) + \sigma_K^2 E(R_K) &= \sigma_K^2 E(R_i) \\
E(R_K)\frac{\sigma_{iK}}{\sigma_K^2} - r\frac{\left(\sigma_{iK} - \sigma_K^2\right)}{\sigma_K^2} &= E(R_i) \\
(E(R_K) - r)\frac{\sigma_{iK}}{\sigma_K^2} + r &= E(R_i)
\end{aligned}$$

CAPM - A mathematical representation

If we denote $\frac{\sigma_{iK}}{\sigma_K^2}$ with β_i we can write

$$E(R_i) = r + \beta_i (E(R_K) - r)$$

The above expression is the β representation of the mean variance optimization problem.

- The expected return on any asset is equal to risk free rate plus a premium risk multiplied by the asset's β

Capital Asset Pricing Model (CAPM)

Capital Market Equilibrium: *demand for risky securities = supply of risky securities*

- Demand for risky securities represented by the **tangency portfolio** (portfolio of all assets demanded on the market)
- Supply of risky assets summarized in **market portfolio**
  - market portfolio is the portfolio of all assets (supplied on the market).
  - portfolio weights: market capitalization of each asset divided by the sum of market capitalization across all assets

Capital Asset Pricing Model (CAPM)

Capital Market Equilibrium: *demand for risky securities = supply of risky securities*

Thus: **in equilibrium market portfolio and tangency portfolio are identical**

Capital Market Line - *the line denoting* **the efficient frontier** *in the CAPM model:*

$$E(R_P) = r + (E_M - r)\frac{\sigma_P}{\sigma_M}$$

$E(R_P)$: expected return of **any portfolio on efficient frontier**

$\frac{E_M - r}{\sigma_M}$: known as the market price of risk

Capital Asset Pricing Model (CAPM)

If we replace **tangency portfolio** with **market portfolio** in the β **representation of mean variance optimization**:

$$E\left(R_i\right) = r + \beta_i \left(E\left(R_M\right) - r\right)$$

CAPM: relation between expected return on **any** asset i and the return on the market

$$E\left(R_i\right) - r = \left(E_M - r\right) \frac{Cov\left(R_i, R_M\right)}{Var\left(R_M\right)}$$

or

$$E\left(R_i\right) - r = \left(E_M - r\right) \beta_i$$

$E_M - r$: **Market Risk Premium**
$\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)}$

Capital Asset Pricing Model (CAPM)

CAPM builds on the insight that “portfolio risk is what matters to investors”

Risk premium on an asset determined by its contribution to investors’ overall portfolio risk

- If $Cov(R_i, R_M)$ is negative, - negative contribution
- If $Cov(R_i, R_M)$ is positive, - positive contribution

Capital Asset Pricing Model (CAPM)

What does "beta" mean?

- how the stock return covaries with the market portfolio return
- $\beta_i = 1$
- implies that the stock return covaries perfectly with the market return
- 1% increase in the market risk premium increases the stock premium by 1%.

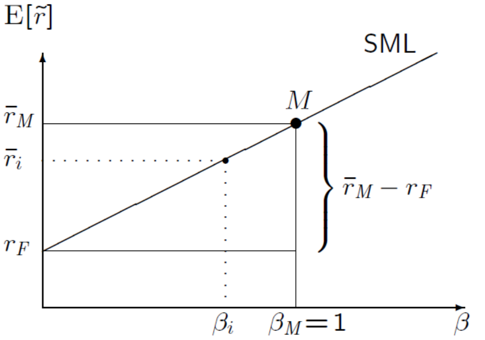
Capital Asset Pricing Model (CAPM)

- $\beta_i > 1$
  - implies that 1% increase in the market risk premium increases the stock premium by more than 1%.
  - these type of stocks are called "high beta" stocks
  - aggressive investments
- $\beta_i < 1$
  - implies that 1% increase in the market risk premium increases the stock premium by less than 1%
  - "low beta" stocks – defensive

Capital Asset Pricing Model (CAPM)

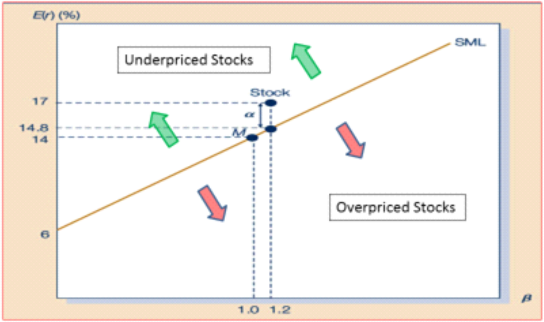
The graphical representation of expected return-beta relationship is given by **Security Market Line**

$$E(R_i) - r = (E_M - r)\beta_i$$



Capital Asset Pricing Model (CAPM)

SML works as a benchmark to assess fair expected return on a risky asset



Alpha of a stock

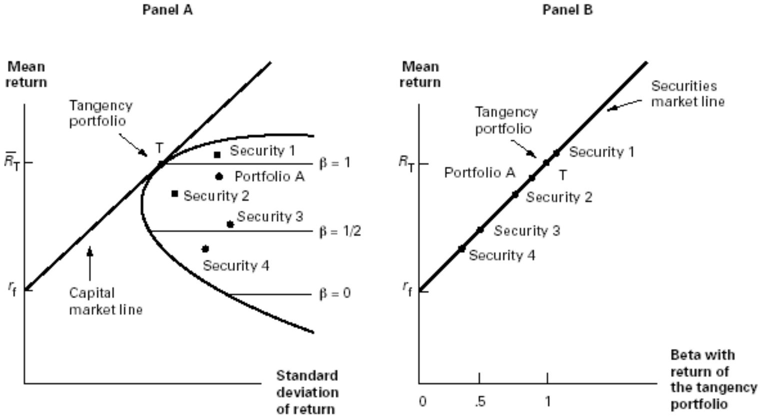
α = actually expected return – required return (given risk) from CAPM

Capital Asset Pricing Model (CAPM)

CML vs *SML*

- *CML* graphs risk premiums of efficient portfolios as a function of portfolio standard deviation
  - standard deviation is a valid measure of risk for efficiently diversified portfolios that are candidates for an investors' overall portfolio.
- *SML* graphs individual asset risk premiums as a function of asset risk, where the appropriate risk measure is the contribution of that asset to the total portfolio risk - the beta

Capital Asset Pricing Model (CAPM)



Systematic Risk and Unsystematic Risk

- A part of the total risk can be eliminated by diversification, i.e. by investing in many different assets
  - This part of total risk that can be eliminated by diversification is called diversifiable risk
- Secondly, there is a minimum level of risk that cannot be eliminated simply by diversifying
  - This part of total risk that cannot be eliminated by diversification is called nondiversifiable risk

Capital Asset Pricing Model (CAPM)

Systematic Risk and Unsystematic Risk

If we hold a large enough portfolio, unsystematic risk of individual companies would cancel each other and only the systematic risk associated with each company will be the risk of the portfolio.

Two types of risk:

- Systematic Risk = Nondiversifiable Risk = Market Risk
  - Systematic risk is common to all stocks and cannot be easily eliminated by diversification
- Diversifiable Risk = Idiosyncratic Risk = Firm-specific Risk = Unsystematic risk
  - Idiosyncratic risk can be easily eliminated by diversification.

Capital Asset Pricing Model (CAPM)

The pros and the cons...

- CAPM: Market Portfolio is mean-variance efficient
- CAPM: Market Equilibrium Mode

Problems:

- Most of the assumptions are unrealistic
- Market portfolio not observable
- Empirical test use broad-based equity index such FTSE-100, S&P500, Nikkei 250 as proxies

$$R_i = r + \beta_i \left(R_{index} - r \right) + \varepsilon_i$$

  - empirical studies do not strongly support the model
- The true market portfolio might contain other financial assets such as bonds and stock that are not in included in this indices as well as non financial assets: (real-estate, human capital).