Gaussian elimination

Definition 6.2.1

A matrix is said to be in *row echelon form* if it satisfies the following three conditions:

(i) All zero rows (consisting entirely of zeros) are at the bottom. ✓

(ii) The first non-zero entry from the left in each nonzero row is a 1, called the *leading* 1 for that row. ✓

(iii) Each leading 1 is to the right of all leading 1's in the rows above it. ✓

A row echelon matrix is said to be in *reduced row echelon form* if, in addition it satisfies the following condition:

(iv) Each leading 1 is the only nonzero entry in its column

$$\text{6 columns}\quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 7 & 8 & 9 & 10 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \rightsquigarrow \frac{1}{7}R_2 \\ R_3 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & \frac{8}{7} & \frac{9}{7} & \frac{10}{7} & \frac{11}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples

Determine the solution set of the following systems:

(a)
$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

2 equations

$$x_1 + 3x_2 + 0 \cdot x_3 = 2$$
$$0x_1 + 0x_2 + 0x_3 = 1 \leftarrow$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S = \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \text{ solves (a)}\}$$
$$= \emptyset \quad \text{incosistent system}$$

(a)'
$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$0x_1 + 0x_2 + 0x_3 = 0$$
$$x_1 + 3x_2 = 2 \Longrightarrow x_1 = 2 - 3x_2$$
$$x_3 = \alpha \in \mathbb{R}$$

$$S = \{(2-3\beta, \beta, \alpha) : \alpha, \beta \in \mathbb{R}\}$$

(b)
$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - 2x_2 + x_4 = 2$$
$$x_3 - 2x_4 = 1$$

$$x_3 = 1 + 2x_4$$
$$x_1 = 2 + 2x_2 - x_4$$

$$S = \{(2+2\alpha-\beta, \alpha, 1+2\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$$

Gaussian algorithm

(1) If the matrix consists entirely of zeros, stop — it is already in row echelon form.

(2) Otherwise, find the first column from the left containing a non-zero entry (call it a), and move the row containing that entry to the top position.

(3) Now multiply that row by $1/a$ to create a leading 1.

$b, c \neq 0$

(4) By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

$R_2 \to R_2 - bR_1$
$R_3 \to R_3 - cR_1$

This completes the first row. All further operations are carried out on the other rows.

(5) Repeat steps 1-4 on the matrix consisting of the remaining rows

The process stops when either no rows remain at Step 5 or the remaining rows consist of zeros.

$$\begin{aligned} x_2 + 6x_3 &= 4 \\ 3x_1 - 3x_2 + 9x_3 &= -3 \\ 2x_1 + 2x_2 + 18x_3 &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 4 \\ 3 & -3 & 9 & -3 \\ 2 & 2 & 18 & 8 \end{array}\right) \quad R_1 \longleftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 9 & -3 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right) \quad R_1 \rightarrow \frac{1}{3} R_1 \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right)$$

$$R_3 \rightarrow R_3 - 2R_1 \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 4 & 12 & 10 \end{array}\right) \quad R_3 \rightarrow R_3 - 4R_2 \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & -12 & -6 \end{array}\right)$$

$$R_3 \rightarrow -\frac{1}{12} R_3 \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\begin{aligned} x_1 &= x_2 - 3x_3 - 1 = 1 - \frac{3}{2} - 1 = \frac{-3}{2} \\ x_2 &= -6x_3 + 4 = -3 + 4 = 1 \\ x_3 &= \frac{1}{2} \end{aligned}$$

$$S = \left\{ \left(-\frac{3}{2}, 1, \frac{1}{2} \right) \right\}$$

Gauss-Jordan algorithm

(1) Bring matrix to row echelon form using the Gaussian algorithm.

(2) Find the row containing the first leading 1 from the right, and add suitable multiples of this row to the rows above it to make each entry above the leading 1 zero.

This completes the first non-zero row from the bottom. All further operations are carried out on the rows above it.

(3) Repeat steps 1-2 on the matrix consisting of the remaining rows.

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 4$$
$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 5$$
$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 7$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 & 2 & 5 \\ 1 & 1 & 1 & 2 & 3 & 7 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$R_1 \to R_1 - R_3$
$R_2 \to R_2 - R_3$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$R_1 \to R_1 - R_2$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$x_5 = 2$ $\quad x_1 + x_2 + x_3 = 3$ $\quad x_2 = \alpha \in \mathbb{R}$
$x_4 = -1$ $\quad x_3 = \beta \in \mathbb{R}$

$$S = \{(3-\alpha-\beta, \alpha, \beta, -1, 2) : \alpha, \beta \in \mathbb{R}\}$$

Conclusion

Theorem 6.2.7

(a) Every matrix can be brought to row echelon form by a series of elementary row operations.

(b) Every matrix can be brought to reduced row echelon form by a series of elementary row operations.

(a) Gaussian algorithm

(b) Gauss - Jordan algorithm

Def A $m \times n$ linear system $m \Big\{ \big(\quad \big)$ with n columns

overdetermined $m > n$

underdetermined $m < n$

Theorem If an underdetermined system is consistent, it must have infinitely many solutions.

Proof This is due to the fact that there are more variables than equations.

Def

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}$$

This system is said to be **homogeneous** if $b_i = 0$ for all i's.

If this is not the case then the system is **inhomogeneous**.

$$\begin{aligned}
3x_1 + 2x_2 + 5x_3 &= 2 \\
2x_1 - x_2 + x_3 &= 5
\end{aligned}$$

inh.

associated hom. system

$$\left|\begin{aligned}
3x_1 + 2x_2 + 5x_3 &= 0 \\
2x_1 - x_2 + x_3 &= 0
\end{aligned}\right|$$

Is it possible that an homogeneous system does not have any solutions? No

Theorem: An underdetermined homogeneous system always has non-trivial solutions.

Proof We have a system of the type

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots - a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= 0 \end{aligned} \qquad m < n$$

Since this system has at least one solution (the trivial one) and it is ~~inconsist.~~ consistent then we have infinite solutions. $\square$

$m = n$?

How many solutions does an homogeneous $n \times n$ system have?

3×3

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

n equations
n variables

Theorem An $n \times n$ system is consistent and has a unique solution iff the only solution of the associated homogeneous system is the zero solution.

Proof

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} & b_n \end{array}\right)$$

- Since the transformations to the augmented matrix are the same for both systems, then if one system has a unique solution the same is true for the other one

- An $n \times n$ system in row echelon form has a unique solution precisely if there are n leading variables.

$\square$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{nn} \end{pmatrix}$$ square matrix