last week

A system of $\underline{\underline{m}}$ linear equation in n unkowns

$$Ax = b$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Coefficient matrix

Augumentedmatrix

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{mn} & b_n \end{array} \right)$$

Gaussian eliminations peform elementary row opertaton augumented matrix to a simpler form to find solutions to the above linear system.

- Row echelon $\Longrightarrow$ easy solutions.

- inconsistent (a system with no solution)
- Consistent (at least one solutions)

- free variable
- leading varable $\longrightarrow$ leading 1.

Example 6.2.4. Determine the solution set of the following system.

no solution a) $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right) \iff \begin{array}{l} x_1 + 3x_2 + 0 \cdot x_3 = 2 \\ 0 = 1 \end{array}$

infinite solutions b) $\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \iff \begin{cases} x_1 - 2x_2 + x_4 = 2 \\ x_3 - 2x_4 = 1 \end{cases}$$

free variables: x_2, x_4

leading variables x_1, x_3

$$x_2 = \alpha$$
$$x_4 = \beta$$

writing down leading variables by free variables

$$x_3 = 1 + 2x_4 = 1 + 2\beta$$
$$x_1 = 2 + 2x_2 - x_4 = 2 + 2\alpha - \beta$$

Thus the solution set for the linear system b) is

$$\{(2+2\alpha-\beta, \alpha, 1+2\beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

Gaussian Algorithm: (GA)

Step 1: If matrix consists entires of zeros. Stop. it is already in row echelon form. (Definition 6.2.1)

Step 2: Otherwise. find the first column from the left containing a non-zero entry (call it a) and move the row containing that entry to the top position.

Step 3: Multiple that row by $1/a$ to create a leading 1

Step 4: By subtracting multiples of that row from rows below it, making each entry below the leading zero.

This completes the first row. All further operation are carried out on the other rows.

Step 5: Repeate Step 1 – 4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at Step 5 or the remaing rows consist of zeros.

Example 6.2.5. Slove the system below using the Gaussian Algorithm

$$\begin{cases} x_2 + 6x_3 = +4 \\ 3x_1 - 3x_2 + 9x_3 = -3 \\ 2x_1 + 2x_2 + 18x_3 = 8 \end{cases}$$

Solution: ① the augmented matrix of the above system is

Round 1

a

Step2: $\left(\begin{array}{ccc|c} 0 & 1 & 6 & 4 \\ 3 & -3 & 9 & -3 \\ 2 & 2 & 18 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & -3 & 9 & -3 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right)$

Step 3 $\xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 18 & 8 \end{array}\right)$ step4 $\xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 4 & 12 & 10 \end{array}\right)$

remaining matrix

Round 2

step4. $\xrightarrow{R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & -1 & +3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & -12 & -6 \end{array}\right)$

Round 3

Step3 $\xrightarrow{R_3 \cdot -\frac{1}{12}} \left(\begin{array}{ccc|c} 1 & -1 & +3 & -1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$

$x_1 \quad x_2 \quad x_3$

4

The corresponding systems of this row echelon is

$$\begin{cases} x_1 - x_2 + 3x_3 = -1 \\ \quad x_2 + 6x_3 = 4 \\ \qquad x_3 = \frac{1}{2} \end{cases}$$

The leading variables are x_1, x_2, x_3, no free variables.

$$x_3 = \frac{1}{2}$$

$$x_2 = 4 - 6x_3 = 4 - 6 \cdot \frac{1}{2} = 1$$

$$x_1 = -1 + x_2 - 3x_3 = -1 + 1 - 3 \cdot \frac{1}{2} = -\frac{3}{2}$$

Thus the system has one unique solution

$$\left\{ \left(-\frac{3}{2}, 1, \frac{1}{2} \right) \right\}$$

Gauss - Jordan Algorithm (GJ)

Step 1 - Bring matrix to row echelon from using the Gaussian algorithm

Step 2:- Finding the row Containing the first leading 1 from the right, and add suitable Multiples of this row to the rows above it to make each entry above the leading 1 zero.

This completes the first non-zero row from the bottom. All further operations are carried out on the rows above it.

Step 3. Repeat steps 1–2 on the matrix consisting of the remaining rows.

The process stops when no rows left in step 3.

Example 6.2.6. Solve the following system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 4 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 5 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 7 \end{cases}$$

Solution: We write down the argumented matrix

GA. Step 4. Round 1

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 & 2 & 5 \\ 1 & 1 & 1 & 2 & 3 & 7 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

remaining matrix GA

GA step 4. Round 2

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Row echelon

GJ step 2 Round 1

$$\xrightarrow[R_2 - R_3]{R_1 - R_3} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The row containing first leading 1 from the right.

remaining matrix in GJ

GJ step 2 Round 2

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Reduced row echelon format

Thus the corresponding equations for the reduced row echelon is

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_4 = -1 \\ x_5 = 2 \end{cases}$$

The solution depends on the free variables x_2, x_3.

and the leading variables are x_1, x_4, x_5

Suppose $x_2 = \alpha$, $x_3 = \beta$

we have $x_1 = 3 - x_2 - x_3 = 3 - \alpha - \beta$.

Thus the solutions are

$$\{ (3-\alpha-\beta, \alpha, \beta, -1, 2) / \alpha, \beta \in \mathbb{R} \}$$

Theorem 6.2.7.

a) Every matrix can be brought to row echelon form by a series of elematritary row operations (GA)

b) Every matrix can be brought to reduce row echelon form by a series of elementary row operations (GJ)

Remark 6.2.8. It can be shown that the reduced row echelon form of a matrix is unique But this doesn't hold for the row echelon form