Vectors & Matrices

Solutions to Problem Sheet 5

1. (i) We take $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{s}$ to be the position vectors of points P, Q and S (respectively). We have

$$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \mathbf{s} - \mathbf{p} \, ,$$
$$\overrightarrow{QS} = \overrightarrow{OS} - \overrightarrow{OQ} = \mathbf{s} - \mathbf{q} \, ,$$

and thus, our orthogonality conditions become

$$(\mathbf{s} - \mathbf{p}) \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 0 \, ,$$
$$(\mathbf{s} - \mathbf{q}) \cdot (6\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}) = 0 \, ,$$

or equivalently

$$\mathbf{s} \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = \mathbf{p} \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \, ,$$
$$\mathbf{s} \cdot (6\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}) = \mathbf{q} \cdot (6\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}) \, .$$

By the definition of position vectors $\mathbf{p}$ and $\mathbf{q}$,

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \, ,$$

and so

$$\mathbf{p} \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = 2 - 5 + 6 = 3 \quad \text{and} \tag{1}$$

(2)

$$\mathbf{q} \cdot (6\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}) = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -10 \\ -2 \end{pmatrix} = 24 - 10 - 10 = 4 \, . \tag{3}$$

Similarly, taking $\mathbf{s} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, we have $\mathbf{s}\cdot(2\mathbf{i}-5\mathbf{j}+\mathbf{k}) = 2x-5y+z$ and $\mathbf{s}\cdot(6\mathbf{i}-10\mathbf{j}-2\mathbf{k}) = 6x-10y-2z$. Combining these formulations with the values computed in (1) and (3) above, we derive the linear system

$$\begin{cases} 2x - 5y + z = 3 \\ 6x - 10y - 2z = 4 \end{cases} . \tag{4}$$

(ii) Our linear system (4) can be represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -5 & 1 & 3 \\ 6 & -10 & -2 & 4 \end{array}\right) .$$

Applying elementary row operations will not change the solution set of this system. We apply a Type III operation to scale the top row by a factor of -3 and add the resulting values to the bottom row, giving

$$\left(\begin{array}{ccc|c} 2 & -5 & 1 & 3 \\ 0 & 5 & -5 & -5 \end{array}\right) .$$

We can perform two Type II operations to scale the top row by a factor of $\frac{1}{2}$ and the bottom row by $\frac{1}{5}$ to get

$$\left(\begin{array}{ccc|c} 1 & -2.5 & 0.5 & 1.5 \\ 0 & 1 & -1 & -1 \end{array}\right) .$$

This matrix is now in row echelon form. To put it in reduced row echelon form, we can perform a final Type III operation to scale the bottom row by a factor of 2.5, and add the resulting values to the top, giving

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right) .$$

The leading variables are x and y, and z is a free variable. By letting $z = \lambda$, we see that all solutions of the system (4) are given by

$$\begin{cases} x = -1 + 2\lambda \\ y = -1 + \lambda \\ z = \lambda \end{cases} .$$

This is the parametric form of a line in three dimensional space. We can rearrange all three equations in this system to write them in terms of λ and equate them, giving the Cartesian formulation

$$\frac{x+1}{2} = y + 1 = z \; .$$

Therefore, the set of all points that satisfy the two orthogonality conditions forms a line in $\mathbb{R}^3$. This makes sense, each orthogonality condition reduced to a linear equation in three dimensions, which is the exact formulation that determines a plane in three dimensional space. We know that the intersection of two planes in three dimensional space gives a line (assuming the planes are not parallel), so all we have done here is evaluate that line.

(iii) As with our derivations of the linear equations in part (i), we find that a point (x, y, z) satisfies this orthogonality condition if and only if it solves the equation

$$8x + 2y - 5z = 16 \; . \tag{5}$$

From part (ii), the first two orthogonality conditions are met for any point (x, y, z) satisfying

$$\begin{cases} x = -1 + 2\lambda \\ y = -1 + \lambda \\ z = \lambda \end{cases} .$$

Hence, in order for a point to satisfy all three orthogonality conditions, it must adhere to both this parametrisation and the equation (5) above. We substitute these formulations for the values x, y and z into (5) to get

$$\begin{aligned} 8x + 2y - 5z &= 8(-1 + 2\lambda) + 2(-1 + \lambda) - 5\lambda \\ &= -10 + 13\lambda \\ &= 16 \ . \end{aligned}$$

Therefore, the only value $\lambda \in \mathbb{R}$ that gives a solution (x, y, z) to all three constraints is $\lambda = 2$. The unique point satisfying these conditions is $(3, 1, 2)$.

(iv) If the final orthogonality condition were replaced by the condition that $\overrightarrow{RS}$ should be orthogonal to the vector $8\mathbf{i} - 15\mathbf{j} - \mathbf{k}$, then as above, we can derive a linear equation equivalent to this constraint. We find that a point (x, y, z) satisifies this modified condition if and only if it solves

$$8x - 15y - z = -61 \ . \tag{6}$$

We need to ensure that any points x, y, z satisfy the first two orthogonality conditions. This is equivalent to a point being attained through the parametrisation given in part (ii). Substituting these formulations for x, y, z into the left-hand side of the equation (6), we get

$$8x - 15y - z = 8(-1 + 2\lambda) - 15(-1 + \lambda) - \lambda = 7 \ .$$

It is clear that the value we receive is not consistent with the right-hand side of (6), and so there does not exist any point (x, y, z) such that all three of these orthogonality conditions are satisfied. The geometric interpretation of this non-existence result is that the line produced from the first two constraints is parallel to the plane identified by the equation (6), and so the intersection between both sets is empty.

2. Suppose we have two planes in three dimensional space, called Π_1 and Π_2. Let Π_1 be identified by the Cartesian equation

$$a_{11}x + a_{12}y + a_{13}z = b_1 \ ,$$

and Π_2 by the Cartesian equation

$$a_{21}x + a_{22}y + a_{23}z = b_2 \ .$$

In order for a point to lie in the intersection $\Pi_1 \cap \Pi_2$, it must solve the linear system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \ , \\ a_{21}x + a_{22}y + a_{23}z = b_2 \ . \end{cases} \tag{7}$$

Thus, if we assume that the intersection of Π_1 and Π_2 consists of *exactly* two distinct points (which we'll call $S_1 = (x_1, y_1, z_1)$ and $S_2 = (x_2, y_2, z_2)$), then S_1 and S_2 provide the only values that solve the linear system (7). Let $S_3 = (x_3, y_3, z_3)$ be defined as the midpoint of S_1 and S_2, we have

$$S_3 = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \ .$$

Substituting the values of S_3 into the first equation of (7) gives

$$\begin{aligned} a_{11}x_3 + a_{12}y_3 + a_{13}z_3 &= a_{11}\left(\frac{x_1 + x_2}{2}\right) + a_{12}\left(\frac{y_1 + y_2}{2}\right) + a_{13}\left(\frac{z_1 + z_2}{2}\right) \\ &= a_{11}\left(\frac{x_1}{2}\right) + a_{12}\left(\frac{y_1}{2}\right) + a_{13}\left(\frac{z_1}{2}\right) + a_{11}\left(\frac{x_2}{2}\right) + a_{12}\left(\frac{y_2}{2}\right) + a_{13}\left(\frac{z_2}{2}\right) \\ &= \frac{1}{2}\Big(a_{11}x_1 + a_{12}y_1 + a_{13}z_1\Big) + \frac{1}{2}\Big(a_{11}x_2 + a_{12}y_2 + a_{13}z_2\Big) \\ &= \frac{1}{2} \cdot b_1 + \frac{1}{2} \cdot b_1 \\ &= b_1 \ , \end{aligned}$$

where middle equality used the fact that both (x_1, y_1, z_1) and (x_2, y_2, z_2) are solutions of the system (7), and therefore solve the first equation. An almost identical result holds for the second equation, and so both equations of the system (7) are satisfied by the values (x_3, y_3, z_3), meaning that the point S_3 also lies in the intersection.

This however, would mean that the intersection has at least three points contained within it, contradicting our stated assumption that it had exactly two. Thus, it is not possible for the intersection of two planes in three dimensional space to contain exactly two points.

3. (i) To determine how each condition can be rewritten in terms of the coefficients of a quadratic, we let $f(x) = ax^2+bx+c$. For the first condition to be satsified, we require $f(-3) = f(1)+20$. We substitute the relevant values directly into our formulation of f to get

$$(9a - 3b + c) = (a + b + c) + 20 \ ,$$

or equivalently,

$$8a - 4b = 20 \ .$$

Hence, this condition reduces to a linear equation. Similarly, the condition that $f(x)$ have a remainder of 2 after division by $(x+1)$ can be expressed as a linear equation. To achieve this, we note that by the Polynomial Remainder Theorem, $f(x)$ will have a remainder of 2 after division by $(x-(-1)$ if and only if $f(-1) = 2$. Substituting the value -1 into our formulation for f, we get

$$a - b + c = 2 \ .$$

Finally, the condition that $f'(2) = 7$ can be written as a linear equation. Since we have defined $f(x)$ to be equal to $ax^2 + bx + c$, then differentiating, this expression, we find $f'(x) = 2ax + b$. Substituting $x = 2$ into this formulation and equating it to 7, we see that the condition reduces to

$$4a + b = 7 \ .$$

The remaining condition cannot be rewritten as a linear equation. Indeed, if we were to complete the square of our formulation of f, we would get

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \ .$$

It is clear from this expression that in order for $f(x)$ to have a minimum value of 5, we would need a to be positive and a, b, c to satisfy the equation

$$-\frac{b^2}{4a} + c = 5 \ .$$

This equation, however, is nonlinear. This can be seen from the first term $\frac{-b^2}{4a}$. The fact that b is taken to a power of two, and that the term includes a ratio of two variables, means that this equation does not comply with our definition of linearity.

(ii) By part (i), a quadratic f satisfies all three linear constraints if and only if its coefficients (a, b, c) are a solution to the linear system

$$\begin{cases} 8a - 4b = 20 \\ a - b + c = 2 \\ 4a + b = 7 \end{cases} .$$

We write this system as the augmented matrix

$$\left(\begin{array}{ccc|c} 8 & -4 & 0 & 20 \\ 1 & -1 & 1 & 2 \\ 4 & 1 & 0 & 7 \end{array}\right) .$$

To solve this system, it will suffice to rewrite this augmented matrix in reduced row echelon form. We start with a Type I operation to swap the first and second rows,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 8 & -4 & 0 & 20 \\ 4 & 1 & 0 & 7 \end{array}\right) ,$$

followed by two Type III operations; subtracting 8 times the first row from the second and subtracting 4 times the first row from the third,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -8 & 4 \\ 0 & 5 & -4 & -1 \end{array}\right) .$$

We now perform a Type II operation to rescale the second row by a factor of $\frac{1}{4}$,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 5 & -4 & -1 \end{array}\right),$$

allowing us to move ahead with another Type III operation, subtracting 5 times the second row from the third,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & -6 \end{array}\right).$$

A final Type II operation, rescaling the third row be a factor of $\frac{1}{6}$, brings the system to row echelon form,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Our final operations will bring the system into reduced row echelon form. We start with a Type III operation to add 2 times the third row to the second,

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

followed by another Type III operation to add -1 times the third row to the first,

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

The third (and final) leading-one of the matrix has only zeros surrounding it in its column. We must now do the same for the second leading-one by performing a Type III operation to add the second row to the first, giving

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) .$$

This system has three leading variables, and no free variables. This guarantees that if a solutions exists, it is unique. In fact, it is clear that the system is solved by taking $a = 2$, $b = -1$ and $c = -1$. Thus, the only quadratic that satisfies all three properties given in the question is

$$f(x) = 2x^2 - x - 1 .$$

Indeed,

$$\begin{aligned} f(-3) = 20 &= f(1) + 20 , \\ f(x) &= (2x-3)(x+1) + 2 , \\ f'(2) &= 7 . \end{aligned}$$

4. Suppose a linear system contained the following pair of equations,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 , \tag{8}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 . \tag{9}$$

(Note that these two equations are labelled as if they are the first two in the system, but this does not affect the generality of our argument. In particular, Type I operations do not change the solution set of a linear system, and so if we wanted to prove this result for any other pair of equations in our system, we could just swap the order of equations until they became the first two in the list.)

Consider the equation formed by taking the coefficients of equation (8), and adding some multiple $\lambda \in \mathbb{R}$ of the coefficients of equation (9),

$$(a_{11} + \lambda a_{21})x_1 + (a_{12} + \lambda a_{22})x_2 + \ldots + (a_{1n} + \lambda a_{2n})x_n = b_1 + \lambda b_2 \ . \tag{10}$$

In order to prove the result, we must show that a set of values $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is a solution to the original linear system if and only if it is a solution to the modified linear system produced when equation (8) is replaced with (10).

We begin by proving the result that all solutions of the original linear system also solve the modified one. Let $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ be a solution to the original system. By definition, this set of values must solve all equations in the system, including (8) and (9). Substituting these values into the left-hand side of (10), we find

$$\begin{aligned}(a_{11} + \lambda a_{21})y_1 + \ldots + (a_{1n} + \lambda a_{2n})y_n &= a_{11}y_1 + \lambda a_{21}y_1 + \ldots + a_{1n}y_n + \lambda a_{2n}y_n \\ &= (a_{11}y_1 + \ldots + a_{1n}y_n) + \lambda(a_{21}y_1 + \ldots + a_{2n}y_n) \\ &= b_1 + \lambda b_2 \ .\end{aligned}$$

Thus, the expression on the left-hand side of (10) equates to the value on the right-hand side, and the values $(y_1, \ldots, y_n)$ also solve equation (10). As $(y_1, \ldots, y_n)$ solves all other equations in the original system, it is a solution to the modified system. We now prove the converse of this argument, that is, all solutions of the modified linear system also solve the original one. Let $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ be a solution to the modified system. We have,

$$\begin{aligned}a_{11}y_1 + \ldots + a_{1n}y_n &= (a_{11} + \lambda a_{21} - \lambda a_{21})y_1 + \ldots + (a_{1n} + \lambda a_{2n} - \lambda a_{2n})y_n \\ &= \Big((a_{11} + \lambda a_{21})y_1 + \ldots + (a_{1n} + \lambda a_{2n})y_n\Big) - \lambda\Big(a_{21}y_1 + \ldots + a_{2n}y_n\Big) \ .\end{aligned}$$

As $(y_1, \ldots, y_n)$ solves equation (10), the expression in the left bracket equates to $b_1 + \lambda b_2$. Similarly, $(y_1, \ldots, y_n)$ solves equation (9), and so the expression in the right bracket equates to b_2. Combining these, we get

$$
\begin{aligned}
a_{11}y_1 + \ldots + a_{1n}y_n &= \Big((a_{11} + \lambda a_{21})y_1 + \ldots + (a_{1n} + \lambda a_{2n})y_n\Big) - \lambda\Big(a_{21}y_1 + \ldots + a_{2n}y_n\Big) \\
&= (b_1 + \lambda b_2) - \lambda b_2 \\
&= b_1 \ .
\end{aligned}
$$

Thus, if $(y_1, \ldots, y_n)$ is a solution to the modified system, the expression on the left-hand side of equation (8) still equates to the value on the right. Hence, $(y_1, \ldots, y_n)$ is a solution of (8). All other equations in the modified system remain unchanged, and so $(y_1, \ldots, y_n)$ is also a solution to the original system.

These two arguments show that $(y_1, \ldots, y_n)$ is a solution of the original system if and only if it is a solution of the modified one. In other words, the solution set of the original system is a subset of the modified system and vice versa, meaning that the solutions sets are equal.