Vectors & Matrices

Problem Sheet 5

1. Let $P = (1, 1, 6)$ and $Q = (4, 1, 5)$ be points in $\mathbb{R}^3$. We choose a point $S \in \mathbb{R}^3$ such that the vector $\overrightarrow{PS}$ is orthogonal $2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, and the vector $\overrightarrow{QS}$ is orthogonal to $6\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}$.

   (i) Let $S = (x, y, z)$. Derive a linear system equivalent to these two conditions on S in terms of the coordinates $x, y, z \in \mathbb{R}$.

   (ii) Write this linear system in reduced row echelon form, and give a geometric interpretation of its solutions.

   (iii) Let $R = (2, 5, 2)$ be a point in $\mathbb{R}^3$. Suppose $\overrightarrow{RS}$ is orthogonal to $8\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$. Find the coordinates of the point S that satisfies this additional constraint, and show that it is unique in doing so.

   (iv) Show that if we changed the third condition so that $\overrightarrow{RS}$ were instead orthogonal to $8\mathbf{i} - 15\mathbf{j} - \mathbf{k}$, there would not exist any points $S \in \mathbb{R}^3$ satisfying all three constraints.

2. Show that it is impossible for the intersection of two planes in three dimensional space to consist of *exactly* two points.

3. Consider the space of all quadratic functions $f(x) = ax^2 + bx + c$. By treating the coefficients of these quadratics as points $(a, b, c) \in \mathbb{R}^3$, it is sometimes possible to rewrite a condition on f as a linear equation in $\mathbb{R}^3$.

   (i) Identify the condition below that *cannot* be expressed as a linear equation:

   - $f(-3) = f(1) + 20$
   - $f(x)$ has a remainder of 2 after division by $(x + 1)$.
   - $f'(2) = 7$.
   - The minimum value of $f(x)$ is 5.

   (ii) Discard the nonlinear condition from (i), and find the unique quadratic f that satisfies all three of the remaining constraints.

4. Prove that performing a Type III elementary row operation on a linear system does not affect its solution set. That is, prove that modifying an equation in a linear system by adding to it a scalar multiple of another equation does not affect the solutions of the system.