MTH5114 **Linear Programming and Games, Spring 2024**

Week 4 Seminar Questions **Viresh Patel**

Warm-Up Question (similar to past exam questions): A set of points $P \subseteq \mathbb{R}^n$ is called *convex* if it satisfies the property that $\mathbf{x} \in P$ and $\mathbf{y} \in P$ implies $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in P$ for all $\lambda \in [0,1]$ (i.e. every convex combination of $\mathbf{x}$ and $\mathbf{y}$ is in P).

Consider an arbitrary linear program in standard equation form. Show that the set of all feasible solutions to this linear program must be a convex set.

Solution: Assume our linear program in standard equation form is given by

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Let $F \subseteq \mathbb{R}^n$ be the set of feasible solutions to the linear program. We must show that F is convex, i.e. we must show that for any $\mathbf{x}, \mathbf{y} \in F$, we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in F$ for all $\lambda \in [0,1]$.

Since $\mathbf{x}$ is feasible, we know $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and since $\mathbf{y}$ is feasible, we know $A\mathbf{y} = \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$. We must check that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is feasible, i.e. we must check that $A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \mathbf{b}$ and that $(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \mathbf{0}$. To see that this is true, note first that

$$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda A\mathbf{x} + (1-\lambda)A\mathbf{y} = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b},$$

as required. Also since $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ we know that $x_i \geq 0$ and $y_i \geq 0$ for each i. Therefore, for each i, we have that $\lambda x_i + (1-\lambda)y_i \geq 0$ for all $\lambda \imath [0,1]$. This implies $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \geq \mathbf{0}$ as required.

Discussion Questions:

1. Look again at the warm up question from the week 3 seminar questions and its solution. There, you were asked to sketch the feasible region of the following linear program.

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 12, \\ & 4x_1 + 2x_2 \leq 12, \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (a) Transform this linear program into standard equation form and write down the corresponding constraint matrix A and vectors, $\mathbf{b}$ and $\mathbf{c}$.

Solution:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 + s_1 = 12, \\ & 4x_1 + 2x_2 + s_2 = 12, \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

We have that

$$\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

(b) For each of the following feasible solutions of the original linear program, find the corresponding feasible solutions of the linear program in standard inequality form and check which of them are basic feasible solutions **(using the definition of basic feasible solution)**. [It will be useful to recap the different ways in which you can show that a set of vectors is linearly independent or linearly dependent by looking at the week 1 seminar questions and/or notes from Linear Algebra I.]

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution: The corresponding feasible solution for the transformed linear program are respectively

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} 3 \\ 0 \\ 6 \\ 0 \end{pmatrix} \qquad \mathbf{w}_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} \qquad \mathbf{w}_4 = \begin{pmatrix} 0 \\ 0 \\ 12 \\ 12 \end{pmatrix}$$

Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ be the four columns of the matrix A, i.e. $\mathbf{c}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ etc.

Note that to check whether a vector is a basic feasible solution, by definition, we must first check that it is feasible, and then check whether the columns of A corresponding to the non-zero entries are linearly independent. The question already tells us that the given vectors are feasible so we need only check the second part.

To decide if $\mathbf{w}_1$ is a basic feasible solutions, we must check whether the columns of A corresponding to the non-zero entries of $\mathbf{w}_1$ are linearly independent. All entries of $\mathbf{w}_1$ are non-zero, so we must check if $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ are linearly independent. These vectors are not independent because we have 4 vectors in $\mathbb{R}^2$ (i.e. more vectors than the dimension of the vector space in which they lie). So $\mathbf{w}_1$ is not a basic feasible solution.

For $\mathbf{w}_2$, we must check whether $\{\mathbf{c}_1, \mathbf{c}_3\}$ are linearly independent. These vectors are linearly independent so $\mathbf{w}_2$ is a basic feasible solution. One way to see that these vectors are linearly independent is to note that the 2×2 matrix you get by putting the vectors together is invertible since it has determinant 0. [Another way to see it is to solve the equations $\lambda\mathbf{c}_1 + \mu\mathbf{c}_3 = \mathbf{0}$ for λ and μ and to show that the only solution is $\lambda = \mu = 0$.]

For $\mathbf{w}_3$, we must check whether $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are linearly independent. Using the same reasoning as for $\mathbf{w}_1$, these vectors are not linearly independent so $\mathbf{w}_3$ is not a basic feasible solution.

For $\mathbf{w}_4$, we must check whether $\{\mathbf{c}_1, \mathbf{c}_2\}$ are linearly independent. These vectors are indeed linearly independent by applying the same reasoning as for $\mathbf{w}_2$. Therefore $\mathbf{w}_4$ is a basic feasible solution.

2. Modify the linear program in Q1 above by adding one more inequality constraint in such a way that there is a basic feasible solution (of the corresponding standard equation form) that has three zero entries. [There are many possible solutions here.]

   Solution: We should add the constraint in such a way that the constraint line passes through one of the extreme point solutions (say $\mathbf{v}_4$) of the original linear program and so that $\mathbf{v}_4$ remains feasible. For example take $x_1 + x_2 \geq 0$. In that case $\mathbf{v}_4$ remains an extreme point solution and by the Theorem at the end of Week 4 [that the basic feasible solutions are precisely the extreme point solutions] $\mathbf{v}_4$ is (still) a basic feasible solution. Moreover it lies on three of the constraint / sign restriction lines, and so the slack is 0 for three of the constraints / sign restrictions. Therefore three of the entries of the corresponding feasible solution of the standard equation form will be zero. (After adding the new constraint, if we look at the new standard equation form, then $\mathbf{v}_4$ now corresponds to the vector $(0, 0, 12, 12, 0)^{\mathsf{T}}$: note that the fifth entry corresponds to the slack variable s_3 that we introduce for our new constraint.)