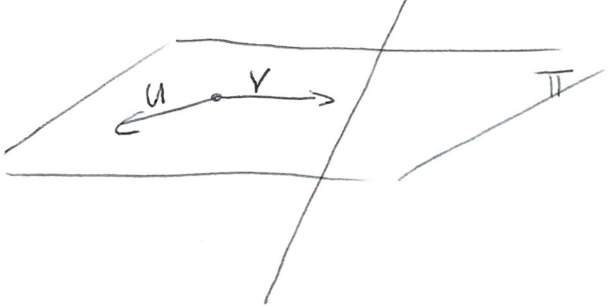
Intersection of other geometric objects

Π



$ax + by + cz = d$ $\qquad n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ normal

$u \times v$ $\qquad \ell$ $\qquad \begin{cases} x = p_1 + \lambda u_1 \\ y = p_2 + \lambda u_2 \\ z = p_3 + \lambda u_3 \end{cases}$ $\quad \lambda \in \mathbb{R}$ parameter

$P \qquad u$

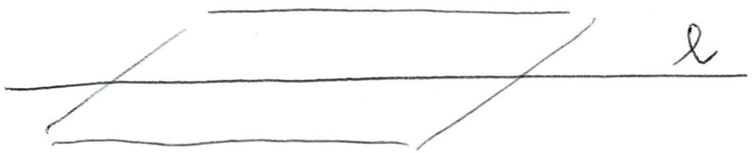
$$a(p_1 + \lambda u_1) + b(p_2 + \lambda u_2) + c(p_3 + \lambda u_3) = d \qquad \lambda \in \mathbb{R}$$

Find $\lambda \in \mathbb{R}$ such that the equation above is fulfilled.

$2\lambda = 3 \implies \lambda = \frac{3}{2}$

$0\lambda = 2$ No solutions

$0\lambda = 0$ infinite solutions



$m \qquad \begin{cases} x = a_1 + \mu v_1 \\ y = a_2 + \mu v_2 \\ z = a_3 + \mu v_3 \end{cases} \qquad \mu \in \mathbb{R}$

$$\begin{cases} a_1 + \mu v_1 = p_1 + \lambda u_1 \\ a_2 + \mu v_2 = p_2 + \lambda u_2 \\ a_3 + \mu v_3 = p_3 + \lambda u_3 \end{cases}$$

Find λ and μ such that the following system is fulfilled

Systems of linear equations

$$\boxed{a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b}$$

First order equation

linear equations

coefficients $a_1, a_2, \dots a_n, b$

unknowns (variables) $x_1, x_2 \dots, x_n$

$x_1, x_2, \dots, x_n$ $\quad \cancel{x_1^2 \quad x_3^4}$

$X^2 = 4 \implies X = \pm 2$

$X^2 = -1$ No solutions in $\mathbb{R}$

$X = \pm i$ 2 solutions in $\mathbb{C}$

$y' + 2y = 0 \quad y = y(x)$

Algebraic equations

Diff. equations

Partial diff. equations

$\partial_t^2 u - \partial_x^2 u = 0 \quad u = u(t, x)$ Wave equation

$$\longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad m \times 1$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

m row

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} = A$$

n col.

$m \times n$ matrix

m rows

n columns

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

$$2x_1 + x_2 = 4$$
$$3x_1 + 2x_2 = 7$$

2 equations
2 variables

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\underline{A x = b} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ \boxed{x_1 + x_2 = 3} \\ 0 \cdot x_1 + x_2 = 1 \end{cases} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\cancel{x_2 = 1}$$
$$\cancel{x_1 = 1}$$

This system has no solutions

Consistent and inconsistent systems

Consistent : if it has at least one solution

inconsistent : it has no solution

Solution set of a system : it is the set of all the solutions of a system

Two systems are equivalent if they have the same solution set

(a)
$$5x_1 - x_2 + 2x_3 = -3$$
$$x_2 = 2$$
$$3x_3 = 6$$

$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$x_3 = 2 \quad x_2 = 2 \quad 5x_1 - 2 + 4 = -3 \quad x_1 = -1$

(b)
$$5x_1 - x_2 + 2x_3 = -3 \quad R_1$$
$$-5x_1 + 2x_2 - 2x_3 = 5 \quad R_2$$
$$5x_1 - x_2 + 5x_3 = 3 \quad R_3$$

$R_2 + R_3 \quad \boxed{x_2 + 3x_3 = 8}$

$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

Lemma

The following operations do not change the solution set of a linear system:

$x_2 = 2 \ \cancel{\Leftrightarrow} \ 0 \cdot x_2 = 2 \cdot 0$

$2x_2 = 4 \Rightarrow x_2 = 2$

(i) interchanging two equations;

(ii) multiplying an equation by a non-zero scalar;

$\lambda, \mu \in \mathbb{R}$

(iii) adding a multiple of one equation to another.

$\lambda R_1 + \mu R_2 \quad \lambda \neq 0, \ \mu \neq 0$

Augmented matrix

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_m \end{array} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

m rows

$n+1$ coloumns

(a) $$\begin{array}{l} 3x_1 + 2x_2 - x_3 = 5 \\ 2x_1 \qquad\quad + x_3 = -1 \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 2 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right)$$

$$R_2 \to 3R_2 - 2R_1 \quad \left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 0 & -4 & 5 & -13 \end{array}\right) \quad (b)$$

Elementary row operations

Type I interchanging two rows;

Type II multiplying a row by a non-zero scalar;

Type III adding a multiple of one row to another row.

$i, j = 1, \dots, m$

$\mu \neq 0$

$R_i + \mu R_j$

(o)

$$3x_1 + 2x_2 - x_3 = 5$$
$$2x_1 + \quad + x_3 = -1$$

Solution set

$$\begin{cases} x_3 = -1 - 2x_1 \\ 3x_1 + 2x_2 + 1 + 2x_1 = 5 \end{cases}$$

$$\begin{cases} x_3 = -2x_1 - 1 \\ 5x_1 + 2x_2 = 4 \end{cases}$$

$$\begin{cases} x_2 = 2 - \frac{5}{2} x_1 \\ x_3 = -2x_1 - 1 \end{cases}$$

$$\begin{cases} x_1 = \lambda \in \mathbb{R} \\ x_2 = -\frac{5}{2}\lambda + 2 \\ x_3 = -2\lambda - 1 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 3 \\ 2x_1 - x_2 + x_3 = 6 \end{cases}$$

$$\begin{array}{cc} R_1 & \multirow{2}{*}{$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -1 & 1 & 6 \end{array}\right)$} \\ R_2 & \end{array}$$

$$R_2 \to R_2 + R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 3 & 0 & 0 & 9 \end{array}\right)$$

$$\begin{cases} x_1 + x_2 - x_3 = 3 \\ 3x_1 \qquad\qquad = 9 \end{cases} \qquad x_1 = 3$$

$$\cancel{3} + x_2 - x_3 = \cancel{3} \qquad \boxed{x_2 = x_3.}$$

$$S = \{ (3, \alpha, \alpha) : \quad \alpha \in \mathbb{R} \}$$